The Geometry of the Narayana Fractal

Jack Farnsworth, Rahul Isaac, and Stella Watson

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Abstract

This paper examines the fractal nature of the Narayana fractal, an object defined by

$$\mathcal{N} = \{(i, j) \in \mathbb{N} \times \mathbb{N} : N(i + j + 1, j + 1) = 1 \pmod 2\}.$$

where

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

are the Narayana numbers. This object closely resembles a fractal derived from Pascal's triangle. This similarity is used to prove that the Hausdorff dimension of the Narayana fractal is $\log 3 / \log 2$, and the limit of the Narayana fractal converges to the union of Sierpinski's gasket with one additional point.

1   Introduction

For each integer $n \geq 0$, let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number. The Catalan numbers arise in a variety of combinatorial problems; indeed, Volume 2 of Stanley's *Enumerative Combinatorics* presents no less than 66 different manifestations of the these numbers. One such manifestation is this: for each integer $n \geq 1$, the number of lattice paths that step only northeast and southeast from $(0, 0)$ to $(2n, 0)$ and do not stray below the x-axis is C_n; see pages 220-229 of [13]. For fixed $n \geq 1$, these Catalan paths can be partitioned according to the number of peaks. Thus, given integers $1 \leq k \leq n$, let $N(n, k)$ denote the number of these paths that contain exactly k peaks. For example, since there are 6 paths from $(0, 0)$ to $(8, 0)$ that meet the
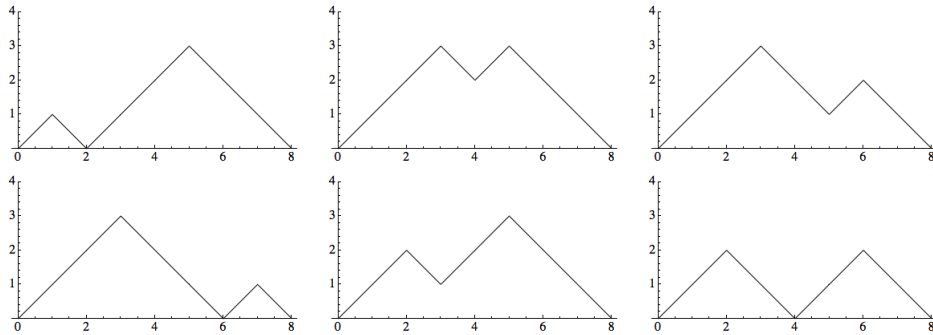
Figure 1: Here are the six paths from $(0,0)$ to $(8,0)$ that contain exactly two peaks.

conditions outlined above and contain exactly 2 peaks, $N(4,2) = 6$. These 6 paths are pictured in Figure 1. Although these numbers were introduced by Percy MacMahon [11], they are presently called the Narayana numbers, in honor of T.V. Narayana who rediscovered them and brought them to prominence [15]. The Narayana numbers have a closed form given by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

and they exhibit an obvious relationship with the Catalan numbers; namely,

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

The Narayana numbers have recently been an object of attention in mathematics and related fields. A 2011 paper by Barry provides two methods for obtaining a generalized form of the Narayana triangle, one using properties of trinomials and one using continued fractions [2]. Li and Mansour present a multiplicative identity for the Narayana numbers [10]. In a subsequent paper, Mansour and Sun show that the Narayana numbers can be written as integrals of the Legendre polynomials [12]. The Narayana numbers have also been used in recent research on MIMO (multiple input, multiple output) communication systems [3]. Our work with the Narayana numbers ties in nicely with a 2005 paper by Bóna and Sagan, in which they give necessary and sufficient conditions for Narayana numbers to be divisible by a given prime [4]. For purposes of self containment, we will not make direct use of
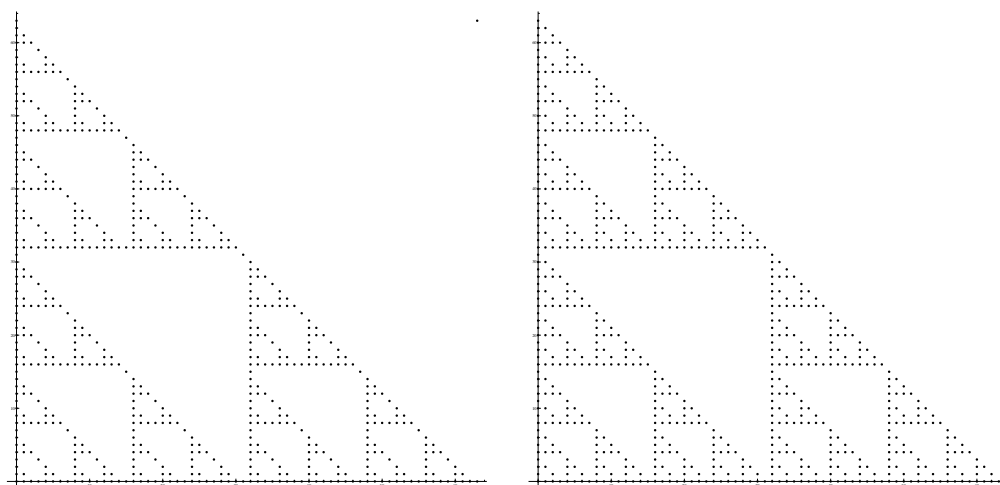
Figure 2: The Narayana fractal (left) and the Pascal fractal (right) modulo 2 in $[0, 63] \times [0, 63]$

their conditions, but their work could provide alternate proofs for the number theoretic lemmas in Section 2.

In this paper, we are concerned with geometric aspects of the Narayana numbers modulo 2. Others have done similar analysis for Pascal's triangle and other number triangles. For example, Wolfram shows that Pascal's triangle modulo 2 is a fractal with Hausdorff dimension $\log 3 / \log 2$ [16]. Sved shows that this result is a special case of a general result for arithmetical functions of two variables satisfying a certain type of recurrence relation[14]. Holte investigates, among other things, the fractal dimension of a set derived from the generalized binomial coefficients modulo a prime [9], and Calvo and Masqué provide a method for calculating the Hausdorff dimension of fractals formed from Pascal's triangle and reduced modulo powers of primes [5]. For general references on fractals, see Edgar [6] and Falconer [7].

Let $\mathbb{N}$ denote the set of nonnegative integers. For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, let

$$\gamma(i, j) = \binom{i+j}{j} \quad \text{and} \quad \nu(i, j) = N(i + j + 1, j + 1).$$

Let the *Pascal fractal* be given by

$$\mathscr{P} = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \gamma(i, j) \equiv 1 \pmod{2}\},$$

and let the *Narayana fractal* be given by

$$\mathcal{N} = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \nu(i, j) \equiv 1 \pmod 2\}.$$

Portions of these fractals are pictured in Figure 2. While these two fractal sets resemble each other on a largest scale, inspection reveals important fine scale differences between them. We will show that despite these fine scale differences, the two fractals are closely related: they have similar patterns of self-propagation, nearly identical limits when re-scaled, and the same discrete Hausdorff dimension as defined by M. T. Barlow and S. J. Taylor [1].

Following, but slightly modifying, the work of Barlow and Taylor, for each $n \geq 0$, let

$$V_n = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j < 2^n\}.$$

We will henceforth refer to V_n as the *window* of size 2^n.

Given $A \subset \mathbb{Z} \times \mathbb{Z}$ and and $x \in \mathbb{Z} \times \mathbb{Z}$, let the *translation of A by x* be given by $A + x = \{a + x : a \in A\}$. Hereafter let

$$\mathcal{P}_n = \mathcal{P} \cap V_n \quad \text{and} \quad \mathcal{N}_n = \mathcal{N} \cap V_n \tag{1.1}$$

for each $n \geq 0$. We are here describing finite portions of the Pascal fractal and the Narayana fractal within a window of size 2^n.

The propagation rule for the Pascal fractal can be stated as follows: for $n \geq 1$

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup \big(\mathcal{P}_n + (0, 2^n)\big) \cup \big(\mathcal{P}_n + (2^n, 0)\big). \tag{1.2}$$

In other words, $\mathcal{P}_{n+1}$ can be formed by joining two copies of $\mathcal{P}_n$ to itself: one copy is translated to the right and the other is translated up.

The propagation rule for the Narayana fractal is similar, but there is a critical difference. For $n \geq 1$, let

$$M_n = \{(0, 2^n - 1), (2^n - 1, 0), (2^n - 1, 2^n - 1)\}.$$

We will call the elements of this set *mortar points*, because they appear to bind the fractal together. We also define

$$\mathcal{N}_n^- = \mathcal{N}_n \setminus M_n$$

Theorem 1.1. *For $n \geq 1$,*

$$\mathcal{N}_{n+1} = \mathcal{N}_n \cup \big(\mathcal{N}_n^- + (0, 2^n)\big) \cup \big(\mathcal{N}_n^- + (2^n, 0)\big) \cup M_{n+1}.$$

It is here that the fine-scale differences between the Pascal fractal and the Narayana fractal emerge. As in the Pascal fractal, $\mathcal{N}_{n+1}$ is formed by joining copies of $\mathcal{N}_n$ to itself. The essential difference is that the nth level mortar is not copied up and over with the rest of the fractal, and new mortar is added at level $n + 1$.

Before we state our next theorem, we will need to introduce some additional notation and make some preparatory observations. As is customary, let $\mathbb{H}([0, 1]^2)$ denote the collection of nonempty, closed subsets of the unit square, $[0, 1]^2$, endowed with the Hausdorff metric. Let

$$M = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix},$$

and, for $x \in [0, 1]^2$, let $F_1(x) = Mx$, $F_2(x) = Mx + (0, 1/2)$, and $F_3(x) = Mx + (1/2, 0)$. Each function is a *contraction* on $[0, 1]^2$ with ratio $1/2$. For $A \subset \mathbb{H}([0, 1]^2)$, let

$$F(A) = F_1(A) \cup F_2(A) \cup F_3(A),$$

which defines a function from $\mathbb{H}([0, 1]^2)$ to $\mathbb{H}([0, 1]^2)$. The function F has a unique fixed point in $\mathbb{H}([0, 1]^2)$, which we will denote by S and identify with Sierpinski's triangle. We will define the iterates of F in the usual way: let $F^{(1)} = F$ and, for $k > 1$, $F^{(k)} = F^{(k-1)} \circ F$. Then for any $A \subset \mathbb{H}([0, 1]^2)$, $F^{(k)}(A) \to S$ in the Hausdorff metric as $k \to \infty$; see, for example, Theorem 8.3 of [7]. In particular, if we set $A = \{(1/2, 1/2)\}$, then

$$\frac{1}{2^n} \left(\mathscr{P}_n + (\tfrac{1}{2}, \tfrac{1}{2}) \right) = F^{(n-1)}(A) \to S$$

in the Hausdorff metric as $n \to \infty$.

For the Narayana fractal, we have a slightly different result. Let $S^+ = S \cup \{(1, 1)\}$.

Theorem 1.2. *As $n \to \infty$,*

$$\frac{1}{2^n} \left(\mathscr{N}_n + (\tfrac{1}{2}, \tfrac{1}{2}) \right) \to S^+$$

in the Hausdorff metric as $n \to \infty$.

The presence of the point $(1,1)$ in the limit reflects the persistence of the mortar.

We will use similarity of $\mathscr{N}$ and $\mathscr{P}$ and the work of Barlow and Taylor to show that the discrete Hausdorff dimension of the Narayana fractal equals that of the Pascal fractal.

Theorem 1.3. $\dim_H(\mathscr{N}) = \log 3/\log 2$.

2 The self-propagation of the Narayana fractal

In this section we will prove Theorem 1.1, which demonstrates how the Narayana fractal propagates from one dyadic window to the next. Our approach emphasizes the geometric relationship between $\mathscr{N}_n$ and $\mathscr{N}_{n+1}$. The main tool of our analysis is Kummer's theorem.

Theorem 2.1 (Kummer's theorem, [8])**.** *The power of the prime p that divides the binomial coefficient $\binom{n+m}{m}$ is the number of carries when adding the p-ary expansions of m and n.*

We will employ the following notation. Given $0 \leq n < 2^k$, the *binary expansion of n* is given by

$$(n)_2 = \varepsilon_{k-1}\varepsilon_{k-2}\cdots\varepsilon_1\varepsilon_0 : \sum_{j=0}^{k-1} \varepsilon_j 2^j = n$$

where $\varepsilon_i \in \{0,1\}$ for each $0 \leq i \leq k-1$. To avoid ambiguity, we will refer to ε_j as the *j*th place of n. We will make frequent use of the following observations: if $2^{k-1} \leq n < 2^k$, then $(n)_2$ has a 1 in position $k-1$; if $(n)_2$ has k trailing 0s, then 2^k divides n.

For $n \geq 1$, let

$$L_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i + j \leq 2^n - 2\}$$
$$U_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 2^n - 1 \leq i + j, \quad i \leq 2^n - 1, \quad j \leq 2^n - 1\}.$$

These sets partition V_n into a lower triangular set, L_n, and an upper triangular set, U_n. Our first three lemmas reveal the relationship between $\mathscr{N}$ and the sets L_n and U_n; they are number-theoretic in nature, leaning heavily on Kummer's theorem.

Lemma 2.2. *For each $n \geq 1$, if $(i, j) \in L_n$, then*

$$\nu(i, j) \equiv \nu(i + 2^n, j) \equiv \nu(i, j + 2^n) \pmod{2}.$$

Proof. Fix $n \geq 1$ and let $(i, j) \in L_n$. We will show $\nu(i, j) \equiv \nu(i + 2^n, j)$ (mod 2); the proof of $\nu(i, j) \equiv \nu(i, j + 2^n)$ (mod 2) follows from the fact that $\nu(i, j) = \nu(j, i)$ for $(i, j) \in \mathbb{N} \times \mathbb{N}$. Thus we need to show that

$$\frac{1}{i + j + 1} \binom{i + j + 1}{j + 1} \binom{i + j + 1}{j}$$

$$\equiv \frac{1}{i + 2^n + j + 1} \binom{i + 2^n + j + 1}{j + 1} \binom{i + 2^n + j + 1}{j} \pmod{2}. \quad (2.3)$$

Let α, β, and γ be the powers of 2 that divide $i + j + 1$, $\binom{i+j+1}{j+1}$, and $\binom{i+j+1}{j}$ respectively, and let A, B, and C be the powers of 2 that divide $i + 2^n + j + 1$, $\binom{i+2^n+j+1}{j+1}$, and $\binom{i+2^n+j+1}{j}$ respectively. We will show that $\alpha = A$, $\beta = B$, and $\gamma = C$. This shows that the same power of 2 divides the left and right sides of equation (2.3), which demonstrates that these two sides are congruent modulo 2. Note that since $(i + (j + 1)) \leq 2^n - 1$, and i and j are nonnegative, $i \leq 2^n - 2$ and $(j + 1) \leq 2^n - 1$.

First we will show that $\alpha = A$. Since $(i, j) \in L_n$, $i + j + 1 \leq 2^n - 1$. So $(i + j + 1)_2$ has a 0 in the nth place. Thus $(i + 2^n + j + 1)_2$ is the same as $(i + j + 1)_2$, but with an additional 1 in the nth place. Thus both have the same number of trailing 0s, which shows that $\alpha = A$.

We will now show that $\beta = B$. By Kummer's theorem, there are β carries in $(i)_2 + (j+1)_2$ and B carries in $(i+2^n)_2 + (j+1)_2$. Note that $(i)_2$, $(j+1)_2$, and $(i + j + 1)_2$ have 0s in place n. Thus $(i + 2^n)_2$ is $(i)_2$ with an additional 1 in place n. Thus $(i)_2 + (j + 1)_2$ and $(i + 2^n)_2 + (j + 1)_2$ have the same number of carries from places 0 to $(n - 1)$. Since $(i + j + 1)$ has a 0 in place n and in all places greater than n, there is no carry from place $(n - 1)$ to place n in $(i)_2 + (j + 1)_2$; therefore, there cannot be a carry from place $(n - 1)$ to place n in $(i + 2^n)_2 + (j + 1)_2$. So there cannot be a carry from place n to $(n + 1)$ in $(i + 2^n)_2 + (j + 1)_2$. Thus $\beta = B$.

Finally we will show $\gamma = C$. By Kummer's theorem, there are γ carries in $(i + 1)_2 + (j)_2$, and C carries in $(i + 2^n + 1)_2 + (j)_2$. Note that both $(i + 1)_2$ and $(j)_2$ have a 0 in place n. So $(i + 2^n + 1)_2$ is $(i + 1)_2$ with an additional 1 in place n. Thus the same number of carries occur in places 0 to $(n - 1)$ in $(i + 1)_2 + (j)_2$ as in $(i + 2^n + 1)_2 + (j)_2$. Since $(i + 2^n)_2$ has a 1 in its nth

place, and $(j)_2$ has a 0 in its nth place, $(i + 2^n + 1)_2 + (j)_2$ does not create a carry from place n to place $(n+1)$. So the number of carries in $(i + 1)_2 + (j)_2$ is the same as the number of carries in $(i + 2^n + 1)_2 + (j)_2$. So $\gamma = C$.

Summarizing our results, we have that the same power of 2 divides both sides of equation (2.3), demonstrating that these two sides are congruent modulo 2, as was to be shown. $\square$

Lemma 2.3. *For each $n \geq 2$, if $(i, j) \in U_n \setminus M_n$ then $\nu(i, j) \equiv 0 \pmod 2$.*

Proof. Fix an integer n, $n \geq 2$. We will divide our proof into two different cases: $i + j > 2^n - 1$ and $i + j = 2^n - 1$.

First assume that $i + j > 2^n - 1$ and let k denote the number of trailing 0s in $(i + j + 1)_2$; we will show that $2^{k+1} | \binom{i+j+1}{j}$, which would show that $\nu(i, j) \equiv 0 \pmod 2$. We will need to consider two subcases: $k \geq 1$ and $k = 0$. If $k \geq 1$, then $(i + 1 + j) \equiv 0 \pmod 2$, implying exactly one of i or j is even. Assume, without loss of generality, that i is even. Consider the binary addition $(i + 1)_2 + (j)_2$. We know that $(i + 1 + j)_2$ has k trailing 0s. However, since both $(i + 1)_2$ and $(j)_2$ are odd, they each have a 1 in place 0; thus, this place creates a carry. Since $(i + 1 + j)_2$ has k trailing 0s, the carried 1 generated in place 0 must carry through at least k places. Note that since the first place is place 0, this implies that the last place to generate a carry is place $k - 1$. If $k = 0$, then, trivially, $(i + 1)_2 + (j)_2$ has k carries. We have thus far counted k carries. We will now prove there is at least 1 additional carry for a total of at least $k + 1$ carries. Since our constraints imply $2^n < i + 1 + j < 2^{n+1} - 1$, we know that k is strictly less than n, and thus we know all carries counted above occur before place n, with place $n - 2$ being the last place that could have created one of our k carries. Note that $(i + 1)_2$ and $(j)_2$ have a 0 in place n and $(i + 1 + j)_2$ has a 1 in place n, so there must exist a carry in from place $n - 1$ to place n. Therefore, there exists $k + 1$ carries in the binary addition $(i + 1)_2 + (j)_2$ and Kummer's theorem implies that $2^{k+1} | \binom{i+j+1}{j}$.

Next, let us assume that $i + j = 2^n - 1$. We will show that $2^{n+1} | \binom{i+1+j}{j}\binom{i+1+j}{j+1}$, which would show that $\nu(i, j) \equiv 0 \pmod 2$. Arguing as above, we can assert that $2^n | \binom{i+1+j}{j}$. However, since there exist n places in $(i + j + 1)_2$, there can exist at most n carries. Note that $(i, j) \in U_n \setminus M_n$ implies $i \neq 0$ and $j \neq 0$. Now consider $(i)_2 + (j + 1)_2$. Since neither i nor j is zero, there must occur a 1 in some position of their binary representations. Since both i and $(j + 1)$ are less than 2^n, and since $(i + j + 1) = 2^n$, the 1 in

$(i)_2$ must either be paired with a 1 in $(j+1)_2$ or a carried 1 in order to sum to a 0. Therefore, there must exist at least one carry in $(i)_2 + (j+1)_2$ and thus, by Kummer's Theorem, $2 \mid \binom{i+j+1}{j+1}$, as was to be shown. $\qquad \square$

Lemma 2.4. $\mathscr{N} \cap M_n = M_n$ for $n \geq 1$.

Proof. Recall that $M_n = \{(0, 2^n - 1), (2^n - 1, 0), (2^n - 1, 2^n - 1)\}$. We will prove that $\nu(i,j) \equiv 1 \pmod 2$ for each $(i,j) \in M_n$. By direct calculation,

$$\nu(0, 2^n - 1) = \nu(2^n - 1, 0) = \frac{1}{2^n} \binom{2^n}{2^n} \binom{2^n}{2^n - 1} = 1.$$

The argument for the third mortar point is more delicate. First note that

$$\nu(2^n - 1, 2^n - 1) = \frac{1}{2^{n+1} - 1} \binom{2^{n+1} - 1}{2^n} \binom{2^{n+1} - 1}{2^n - 1}.$$

Since the addition $(2^n)_2 + (2^n - 1)_2$ creates no carries, an application of Kummer's theorem shows that 2 does not divide $\binom{2^{n+1}-1}{2^n}$ or $\binom{2^{n+1}-1}{2^n-1}$. Thus $\nu(2^n - 1, 2^n - 1) \equiv 1 \pmod 2$. So $\mathscr{N} \cap M_n = M_n$, as was to be shown. $\qquad \square$

Before we turn to the proof of Theorem 1.1, we will establish two corollaries.

Corollary 2.5. *For each* $n \geq 1$,

$$\mathscr{N} \cap L_{n+1} = \mathscr{N}_n \cup \left(\mathscr{N} \cap L_n + (0, 2^n) \right) \cup \left(\mathscr{N} \cap L_n + (2^n, 0) \right)$$

Proof. First we will show that for each $n \geq 1$,

$$\mathscr{N} \cap L_n + (0, 2^n) = \mathscr{N} \cap (L_n + (0, 2^n)) \tag{2.4}$$

and

$$\mathscr{N} \cap L_n + (2^n, 0) = \mathscr{N} \cap (L_n + (2^n, 0)). \tag{2.5}$$

We will prove only equation (2.4); the proof of equation (2.5) is similar. Let $(i,j) \in \mathscr{N} \cap L_n + (0, 2^n)$. Then $\nu(i, j - 2^n) \equiv 1 \pmod 2$ and $(i, j - 2^n) \in L_n$. But, by Lemma 2.2, it follows that $\nu(i,j) \equiv 1 \pmod 2$, and, since $(i,j) \in L_n + (0, 2^n)$, we may conclude that $(i,j) \in \mathscr{N} \cap (L_n + (0, 2^n))$; hence,

$$\mathscr{N} \cap L_n + (0, 2^n) \subset \mathscr{N} \cap (L_n + (0, 2^n)).$$

A similar line of reasoning shows that $\mathcal{N} \cap L_n + (0, 2^n) \supset \mathcal{N} \cap (L_n + (0, 2^n))$, which verifies (2.4).

To finish our proof, observe that L_{n+1} can be partitioned as

$$L_{n+1} = V_n \cup \left(L_n + (0, 2^n)\right) \cup \left(L_n + (2^n, 0)\right).$$

It follows that $\mathcal{N} \cap L_{n+1}$ can be expressed as

$$\left(\mathcal{N} \cap V_n\right) \cup \left(\mathcal{N} \cap \left(L_n + (0, 2^n)\right)\right) \cup \left(\mathcal{N} \cap \left(L_n + (2^n, 0)\right)\right).$$

By definition, $\mathcal{N}_n = \mathcal{N} \cap V_n$; the rest of the proof follows from equations (2.4) and (2.5). $\qquad\square$

Corollary 2.6. *For each $n \geq 1$, $\mathcal{N} \cap U_n = M_n$.*

Proof. Note that $U_1 = M_1 = \{(0, 1), (1, 0), (1, 1)\}$, and that $\nu(0, 1) \equiv \nu(1, 0) \equiv \nu(1, 1) \equiv 1 \pmod 2$. For $n \geq 2$, observe that

$$\mathcal{N} \cap U_n = [\mathcal{N} \cap (U_n \setminus M_n)] \cup (\mathcal{N} \cap M_n).$$

By Lemma 2.3, $\mathcal{N} \cap (U_n \setminus M_n) = \varnothing$ and, by Lemma 2.4, $\mathcal{N} \cap M_n = M_n$, which proves our claim for each $n \geq 2$. $\qquad\square$

We will now prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 2.5 we have:

$$\mathcal{N} \cap L_{n+1} = \mathcal{N}_n \cup \left(\mathcal{N} \cap L_n + (0, 2^n)\right) \cup \left(\mathcal{N} \cap L_n + (2^n, 0)\right).$$

By Corollary 2.6, we have $\mathcal{N} \cap U_{n+1} = M_{n+1}$. Since

$$\mathcal{N} \cap V_{n+1} = (\mathcal{N} \cap L_{n+1}) \cup (\mathcal{N} \cap U_{n+1}),$$

we can conclude that

$$\mathcal{N}_{n+1} = \mathcal{N}_n \cup \left(\mathcal{N} \cap L_n + (0, 2^n)\right) \cup \left(\mathcal{N} \cap L_n + (2^n, 0)\right) \cup M_{n+1}.$$

Since $\mathcal{N} \cap L_n = \mathcal{N}_n^-$, it follows that

$$\mathcal{N}_{n+1} = \mathcal{N}_n \cup \left(\mathcal{N}_n^- + (0, 2^n)\right) \cup \left(\mathcal{N}_n^- + (2^n, 0)\right) \cup M_{n+1},$$
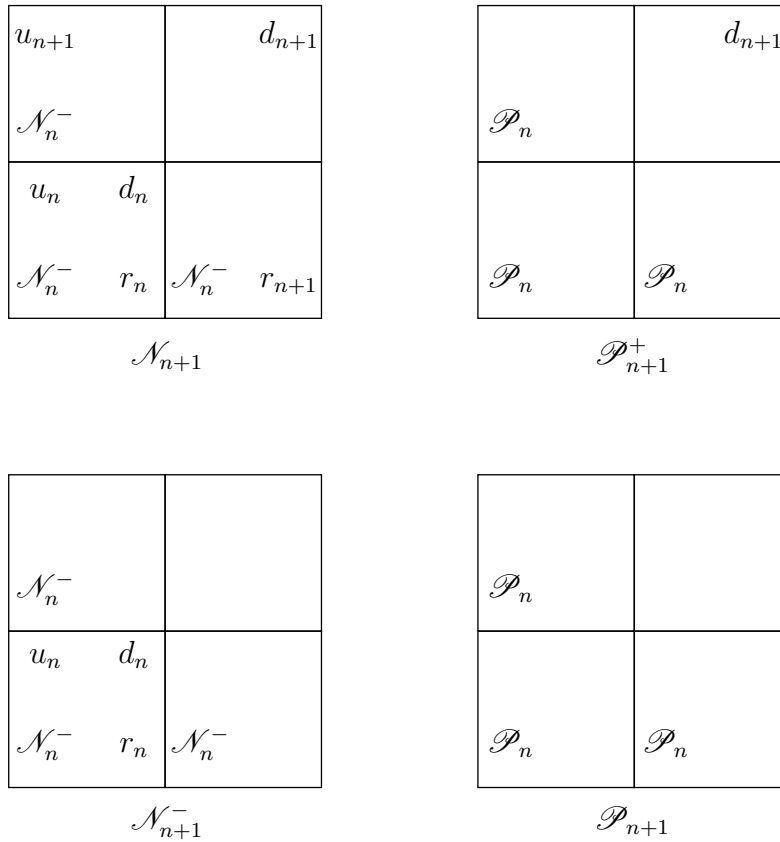
as was to be shown. $\qquad\square$

Figure 3: This diagram shows how each of the four sets is related to its constituent parts.

3 The convergence of the Narayana fractal

For each $n \in \mathbb{N}$, let $u_n = (2^n - 1, 0)$, $d_n = (2^n - 1, 2^n - 1)$, and $r_n = (0, 2^n - 1)$. It is helpful to recall that these are the three elements of the set M_n. We have used the labels u, d, and r to suggest the orientation of these points within V_n: namely, up, diagonal, and right. For each $n \in \mathbb{N}$, let

$$\mathscr{P}_n^+ = \mathscr{P}_n \cup \{(2^n - 1, 2^n - 1)\} = \mathscr{P}_n \cup \{d_n\}.$$

Our next theorem asserts that the sets $\mathscr{P}_n^+$ and $\mathscr{N}_n$ are close to each other for each $n \in \mathbb{N}$. Given sets $A, B \subset \mathbb{Z} \times \mathbb{Z}$, we say that A and B are *proximal* provided that for each $a = (a_1, a_2) \in A$ there exists $b = (b_1, b_2) \in B$ such that $|a_1 - b_1| + |a_2 - b_2| \le 1$ and for each $b = (b_1, b_2) \in B$ there exists

$a = (a_1, a_2) \in A$ such that $|a_1 - b_1| + |a_2 - b_2| \leq 1$. We will write $A \sim B$ whenever A and B are proximal.

Theorem 3.1. *For each $n \geq 1$, $\mathcal{N}_n \sim \mathcal{P}_n^+$ and $\mathcal{N}_n^- \sim \mathcal{P}_n$.*

Proof. We will proceed by induction on n. First observe that

$$\mathcal{N}_1 = \mathcal{P}_1^+ = \{(0,0), (1,0), (0,1), (1,1)\};$$

thus, $\mathcal{N}_1 \sim \mathcal{P}_1^+$ trivially. Likewise,

$$\mathcal{N}_1^- = \{(0,0)\} \quad \text{and} \quad \mathcal{P}_1 = \{(0,0), (1,0), (0,1)\};$$

thus, $\mathcal{N}_1^- \sim \mathcal{P}_1$ by inspection.

Let us assume that the claim is true for some n. We will show only that $\mathcal{N}_{n+1} \sim \mathcal{P}_{n+1}^+$; the proof of $\mathcal{N}_{n+1}^- \sim \mathcal{P}_{n+1}$ is nearly identical.

Let the window V_{n+1} be partitioned into 4 quadrants as follows:

$$Q_1 = V_n, \; Q_2 = V_n + (0, 2^n), \; Q_3 = V_n + (2^n, 2^n), \; Q_4 = V_n + (2^n, 0). \quad (3.6)$$

We will examine the claim for $\mathcal{N}_{n+1}$ and $\mathcal{P}_{n+1}^+$ within each of these quadrants. In Figure 3, we have a diagram which shows how each of the four sets in question is related within these quadrants to its constituent parts. The heavy lifting in the proof of this theorem is done by the propagation rules for each fractal, Theorem 1.1 and equation (1.2). By way of these propagation rules, we have

$$\mathcal{N}_{n+1} \cap Q_i \sim \mathcal{P}_{n+1}^+ \cap Q_i \quad \text{for } 2 \leq i \leq 4. \quad (3.7)$$

The exceptional case is in Q_1, owing to the fact that the point d_n, which is an element of $\mathcal{N}_{n+1}$, is no longer present in $\mathcal{P}_{n+1}^+$.

Consider $\mathcal{N}_{n+1}$ and $\mathcal{P}_{n+1}^+$ within the set Q_1. By induction, $\mathcal{P}_n \sim \mathcal{N}_n^-$. Since $u_n, r_n \in \mathcal{P}_n$, in fact $\mathcal{N}_n^- \cup \{u_n, r_n\} \sim \mathcal{P}_n$. At this point, we meet an impasse, since there is no point in $\mathcal{P}_n$ which is adjacent to d_n; however, $(2^n - 1, 2^n) = d_n + (0, 1)$ is an element of $\mathcal{P}_{n+1}$; therefore, d_n is adjacent to an element of $\mathcal{P}_{n+1}$. Consider $\mathcal{N}_{n+1}$ and $\mathcal{P}_{n+1}^+$ within the set Q_2. By induction, the copy of $\mathcal{N}_n^-$ in Q_2 of $\mathcal{N}_{n+1}$ is proximal to the copy of $\mathcal{P}_n$ within Q_2 of $\mathcal{P}_{n+1}^+$. Since $u_{n+1} \in \mathcal{P}_{n+1}$, in fact $\mathcal{N}_{n+1} \cap Q_2 \sim \mathcal{P}_{n+1}^+ \cap Q_2$. The same argument shows that $\mathcal{N}_{n+1} \cap Q_4 \sim \mathcal{P}_{n+1}^+ \cap Q_4$. Since $\mathcal{N}_{n+1} \cap Q_3 = \mathcal{P}_{n+1}^+ \cap Q_3 = d_{n+1}$, clearly $\mathcal{N}_{n+1} \cap Q_3 \sim \mathcal{P}_{n+1}^+ \cap Q_3$. In summary, $\mathcal{N}_{n+1} \sim \mathcal{P}_{n+1}^+$, as was to be shown. $\square$

Let ρ_H denote the usual Hausdorff metric; see, for example, page 71 of [6]. We can easily recast Theorem 3.1 into the language of Hausdorff metric.

Corollary 3.2. *For each* $n \in \mathbb{N}$, $\rho_H(\mathscr{N}_n, \mathscr{P}_n^+) \leq 1$.

We are now prepared to prove Theorem 1.2.

Proof of Theorem 1.2. By the triangle inequality,

$$\rho_H\left(\frac{1}{2^n}\left(\mathscr{N}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), S^+\right)$$

$$\leq \rho_H\left(\frac{1}{2^n}\left(\mathscr{N}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), \frac{1}{2^n}\left(\mathscr{P}_n^+ + (\tfrac{1}{2}, \tfrac{1}{2})\right)\right) + \rho_H\left(\frac{1}{2^n}\left(\mathscr{P}_n^+ + (\tfrac{1}{2}, \tfrac{1}{2})\right), S^+\right)$$

By Corollary 3.2 and scaling,

$$\rho_H\left(\frac{1}{2^n}\left(\mathscr{N}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), \frac{1}{2^n}\left(\mathscr{P}_n^+ + (\tfrac{1}{2}, \tfrac{1}{2})\right)\right) \leq \frac{1}{2^n}.$$

Observe that

$$\rho_H\left(\frac{1}{2^n}\left(\mathscr{P}_n^+ + (\tfrac{1}{2}, \tfrac{1}{2})\right), S^+\right) \leq \rho_H\left(\frac{1}{2^n}\left(\mathscr{P}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), S\right) + \frac{1}{2^n}$$

By the definition of S (see Section 1),

$$\rho_H\left(\frac{1}{2^n}\left(\mathscr{P}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), S\right) \to 0$$

as $n \to \infty$. Summarizing our findings, as $n \to \infty$,

$$\rho_H\left(\frac{1}{2^n}\left(\mathscr{N}_n + (\tfrac{1}{2}, \tfrac{1}{2})\right), S^+\right) \to 0,$$

as was to be shown. $\square$

4   The dimension of the Narayana fractal

Throughout this section we will follow Barlow and Taylor's development of the Hausdorff dimension of a discrete fractal; see [1]. Before entering into the heart of the proof of Theorem 1.3, we will briefly review their work. It should be noted here that Barlow and Taylor's treatment of Hausdorff dimension is more general, considering subsets of $\mathbb{Z}^n$; our presentation is an adaptation of their work to subsets of $\mathbb{N} \times \mathbb{N}$.

The definition of the discrete Hausdorff dimension requires some special subsets of $\mathbb{N} \times \mathbb{N}$, called shells and cubes. Let $S_0 = \{(0,0)\}$ and, for $n \geq 1$, let $S_n = V_n \setminus V_{n-1}$; each such set is called a *shell*. Given $(x_0, y_0) \in \mathbb{N} \times \mathbb{N}$ and $k \in \mathbb{N}$, the *cube* of width 2^k anchored at (x_0, y_0), denoted by $C((x_0, y_0), 2^k)$, is

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} : x_0 \leq x < x_0 + 2^k \text{ and } y_0 \leq y < y_0 + 2^k\}.$$

Given a cube C, we will write $|C|$ to denote its width, that is, the number of points on its side. In particular, $|C((x_0, y_0), 2^k)| = 2^k$.

A *covering* of a set $A \subset \mathbb{N} \times \mathbb{N}$ is a collection of cubes, not necessarily of the same width, whose union contains A. Let $\alpha > 0$ and let the set of cubes $\{C((x_i, y_i), 2^{k_i}) : 1 \leq i \leq m\}$ cover the set $A \cap S_n$. The α-*cost* of this covering is

$$\sum_{i=1}^{m} (|C((x_i, y_i), 2^{k_i})|/2^n)^\alpha = \sum_{i=1}^{m} (2^{k_i}/2^n)^\alpha.$$

Let $\eta_\alpha(A, n)$ be the minimum α-cost taken over all coverings of the set $A \cap S_n$ and set $m_\alpha(A) = \sum_{n=0}^{\infty} \eta_\alpha(A, n)$. The (discrete) *Hausdorff dimension* of A is

$$\dim_H(A) = \inf \{\alpha > 0 : m_\alpha(A) < \infty\}.$$

Given a cube $C = C((x_0, y_0), 2^k)$, define the set C' of five cubes as follows:

$$C' = \{C((x_0, y_0), 2^k), C((x_0 + 2^k, y_0), 2^k), C((x_0 - 2^k, y_0), 2^k),$$
$$C((x_0, y_0 + 2^k), 2^k), C((x_0, y_0 - 2^k), 2^k)\}.$$

Thus C' comprises the original cube plus four translations of that cube, one in each of the four principle directions, up, down, right, and left. Given a set of cubes $R = \{C_i : 1 \leq i \leq m\}$, define the set of cubes R' as follows:

$$R' = \cup_{i=1}^{m} C_i'.$$

A key observation regarding the collection of cubes R' is given in our next lemma. The proof follows trivially from the definition of proximal given in Section 3 and will be omitted.

Lemma 4.1. *Let A and B be proximal subsets $\mathbb{N} \times \mathbb{N}$. If a set of cubes R covers A, then the set of cubes R' covers B.*

We will now prove Theorem 1.3.

Proof of Theorem 1.3. Throughout we will use the fact that $\mathcal{N} \cap S_n = \mathcal{N}_n \cap S_n$ and $\mathcal{P} \cap S_n = \mathcal{P}_n \cap S_n$; see equation (1.1).

The real work in our proof is to establish the following two bounds: let $\alpha > 0$ be fixed; for each $n \geq 0$,

$$\eta_\alpha(\mathcal{N}, n) \geq \eta_\alpha(\mathcal{P}, n)/5 \tag{4.8}$$

and

$$\eta_\alpha(\mathcal{P}, n) \geq \eta_\alpha(\mathcal{N}, n)/5 - 1/2^{n\alpha}. \tag{4.9}$$

The rest of the proof follows easily from this. From inequality (4.8), we can conclude that $m_\alpha(\mathcal{N}) \geq m_\alpha(\mathcal{P})/5$ hence $\dim_H(\mathcal{N}) \geq \dim_H(\mathcal{P})$. Likewise, from inequality (4.9), we can conclude that

$$m_\alpha(\mathcal{P}) \geq \frac{1}{5} m_\alpha(\mathcal{N}) - \frac{2^\alpha}{2^\alpha - 1}$$

hence $\dim_H(\mathcal{P}) \geq \dim_H(\mathcal{N})$. It follows that $\dim_H(\mathcal{N}) = \dim_H(\mathcal{P})$, which proves our claim, since $\mathcal{P}$ has Hausdorff dimension $\log(3)/\log(2)$ [16].

To prove inequality (4.8), let $n \geq 0$ be given and let $R = \{C_i : 1 \leq i \leq m\}$ be a covering of the set $\mathcal{N} \cap S_n = \mathcal{N}_n \cap S_n$ by cubes. From equation (3.7), it follows that $\mathcal{N}_n \cap S_n \sim \mathcal{P}_n^+ \cap S_n$; thus, by Lemma 4.1, R' covers $\mathcal{P} \cap S_n$. Hence

$$\sum_{C_i \in R} (|C_i|/2^n)^\alpha = \frac{1}{5} \sum_{C_i \in R} 5(|C_i|/2^n)^\alpha = \frac{1}{5} \sum_{\Gamma_i \in R'} (|\Gamma_i|/2^n)^\alpha \geq \eta_\alpha(\mathcal{P}, n)/5.$$

This verifies inequality (4.8).

Likewise, to prove inequality (4.9), let $n \geq 0$ be given and let $\{C_i : 1 \leq i \leq m\}$ be a covering of $\mathcal{P} \cap S_n = \mathcal{P}_n \cap S_n$. Recall that $\mathcal{P}_n^+ = \mathcal{P}_n \cup \{d_n\}$;

thus the collection $\{C_i : 1 \leq i \leq m\} \cup \{C(d_n, 2^0)\}$ yields a cover of $\mathscr{P}_n^+ \cap S_n$. This shows that

$$\eta_\alpha(\mathscr{P}_n^+, n) \leq (1/2^n)^\alpha + \sum_{i=1}^{m} (|C_i|/2^n)^\alpha.$$

Since this is true for every cover of $\mathscr{P} \cap S_n$, $\eta_\alpha(\mathscr{P}_n^+, n) - 1/2^{n\alpha} \leq \eta_\alpha(\mathscr{P}, n)$.

Now let $R = \{C_i : 1 \leq i \leq m\}$ be a covering of $\mathscr{P}_n^+ \cap S_n$ by cubes. Then, arguing as above, R' is a cover of $\mathscr{N}_n \cap S_n$; thus,

$$\sum_{C_i \subset R} (|C_i|/2^n)^\alpha = \frac{1}{5} \sum_{C_i \subset R} 5(|C_i|/2^n)^\alpha = \frac{1}{5} \sum_{\Gamma_i \subset R'} (|\Gamma_i|/2^n)^\alpha \geq \eta_\alpha(\mathscr{N}, n)/5.$$

From this we obtain $\eta_\alpha(\mathscr{P}_n^+, n) \geq \eta_\alpha(\mathscr{N}, n)/5$. In summary,

$$\eta_\alpha(\mathscr{P}, n) \geq \frac{1}{5}\eta_\alpha(\mathscr{N}, n) - 1/2^{n\alpha},$$

which gives inequality (4.9), completing our proof. $\qquad\square$

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