QUADRATIZATION OF ODEs: MONOMIAL VS. NON-MONOMIAL

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ABSTRACT. Quadratization is a transform of a system of ODEs with polynomial right-hand side into a system of ODEs with at most quadratic right-hand side via the introduction of new variables. It has been recently used as a preprocessing step for new model order reduction methods, so it is important to keep the number of new variables small. Several algorithms have been designed to search for a quadratization with the new variables being monomials in the original variables. To understand the limitations and potential ways of improving such algorithms, we study the following question: can quadratizations with not necessarily new monomial variables produce a model of substantially smaller dimension than quadratization with only new monomial variables?

To do this, we restrict our attention to scalar polynomial ODEs. Our first result is that a scalar polynomial ODE $\dot{x} = p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $n \geqslant 5$ and $a_n \neq 0$ can be quadratized using exactly one new variable if and only if $p(x - \frac{a_{n-1}}{n \cdot a_n}) = a_n x^n + ax^2 + bx$ for some $a, b \in \mathbb{C}$. In fact, the new variable can be taken as $z := (x + \frac{a_{n-1}}{n \cdot a_n})^{n-1}$. Our second result is that two new non-monomial variables are enough to quadratize all degree 6 scalar polynomial ODEs. Based on these results, we observe that a quadratization with not necessarily new monomial variables can be much smaller than a monomial quadratization even for scalar ODEs.

The main results of the paper have been discovered using computational methods of applied nonlinear algebra (Gröbner bases), and we describe these computations.

Keywords: quadratization, nonlinear ODEs, model order reduction, Gröbner bases, symbolic computation

1. Introduction

Model order reduction of non-linear dynamical systems is an important tool in applied mathematics. The goal of reducing dynamical systems is to make them easier to analyze. In this paper, we investigate quadratization, a technique used as a preprocessing step to some recent model order reduction methods.

Quadratization is a transformation of a system of ordinary differential equations (ODEs) with polynomial right-hand side into a system with at most quadratic right-hand side via the introduction of new variables. While quadratization does indeed lift up the dimension of a system, there are more powerful model order reduction methods for ODEs with at most quadratic right-hand side that produce better reductions than general methods [4, 7]. In essence, quadratization lifts up the dimension of a system for it be pulled down by dedicated model order reduction methods.

We illustrate quadratization using the following simple scalar polynomial ODE:

$$\dot{x} = x^{10} \tag{1.1}$$

The right-hand side has degree greater than two, but by introducing $z := x^9$, we can write:

$$\begin{cases} \dot{x} = zx \\ \dot{z} = 9x^8\dot{x} = 9x^{18} = 9z^2 \end{cases} \tag{1.2}$$

In (1.2), the right-hand side of $\dot{x}$ and $\dot{z}$ are at most quadratic. Furthermore, every solution of (1.1) yields a solution of (1.2). We say that the order of a quadratization is the number of new variables introduced. Thus, this quadratization has order 1.

In (1.2), our new variable z is a monomial in x. Thus, we call this a monomial quadratization. On the other hand, if one of our new variables has non-monomial right-hand side, we will refer to this as non-monomial quadratization. Several algorithms have been developed to find monomial quadratizations of ODEs and ODE systems [1, 2, 6]. An additional motivation to study monomial quadratization comes from the desire to create realistic chemical reaction networks (CRNs), which can be interpreted as polynomial ODEs [6]. For other applications of quadratization, see [5].

However, we are not aware of any algorithms for finding an optimal quadratization with not necessarily new monomial variables. In order to understand the potential benefits of such algorithms, we ask the following question: can non-monomial quadratizations produce a system of substantially smaller dimension than monomial quadratizations? Based on our results, the answer is yes.

In order to approach this question, we consider the case of scalar polynomial ODEs as it is "the simplest nontrivial case." While model order reduction techniques are not useful for scalar polynomial ODEs as their dimension is already minimized at 1, they allow us a point of entry to understand the question posed above.

In our research, we completely characterized the case when one new variable is enough to quadratize a scalar polynomial ODE (see Theorem 3.1). We also found that any degree 6 scalar polynomial ODE can be quadratized with two new non-monomial variables and we give the form of the new variables and quadratization (see Theorem 3.2).

We also show that all degree 3 and 4 scalar polynomial ODEs can be quadratized with one new monomial variable and all degree 5 scalar polynomial ODEs can be quadratized with two new monomial variables (see Proposition 3.1). Each of the quadratizations presented in our main results are optimal, meaning they introduce as few variables as possible.

In order to achieve these results, we employed computational techniques that made use of Gröbner basis (see section 6). In Section 5, we provide mathematical proofs of Theorem 3.1 and Theorem 3.2.

2. Preliminaries

Definition 2.1. Consider the following scalar polynomial ODE:

$$\dot{x} = p(x) \tag{2.1}$$

where $p(x) \in \mathbb{C}[x]$. Then, a list of m new variables:

$$z_1 := z_1(x), z_2 := z_2(x), \ldots, z_m := z_m(x) \tag{2.2}$$

is said to *quadratize* $\dot{x}$ if there exist polynomials $h_1, h_2, \ldots, h_{m+1} \in \mathbb{C}[x, z_1, z_2, \ldots, z_m]$ of degree at most two such that:

- $\dot{x} = h_1(x, z_1, z_2, \ldots, z_m)$;
- $\dot{z}_i = z_i'(x)\dot{x} = h_{i+1}(x, z_1, z_2, \ldots, z_m)$ for $1 \leqslant i \leqslant m$

The number m is said to be the *order* of the quadratization. A quadratization of the smallest possible order is called *optimal*. We refer to new variables whose right-hand sides are monomials as *new monomial variables* and those whose right-hand sides are not monomials as *new non-monomial variables*.

Example 2.1. Consider the scalar polynomial ODE $\dot{x} = x^n$ with $n > 2$. Let $z := x^{n-1}$. Note that z is a new monomial variable. We will use z to quadratize $\dot{x}$. We can write:

$$\begin{cases} \dot{x} = zx \\ \dot{z} = z'(x)\dot{x} = (n-1)x^{n-2} \cdot x^n = (n-1)x^{2n-2} = (n-1)z^2 \end{cases} \tag{2.3}$$

Thus, we have quadratized $\dot{x}$ with $z := x^{n-1}$ as both $\dot{x}$ and $\dot{z}$ can be written as quadratic polynomials in z and x. In particular, this quadratization has order 1 and is optimal.

Example 2.2. Consider the scalar polynomial ODE $\dot{x} = x^5 + x^4 + x^3 + x^2 + x + 1$. We let $z_1(x) := x^4$ and $z_2(x) := x^3$. Note that z_1 and z_2 are new monomial variables. It follows that:

$$\begin{cases} \dot{x} = z_1 x + z_1 + z_2 + x^2 + x + 1 \\ \dot{z}_1 = z_1'(x)\dot{x} = 4x^3\dot{x} = 4(z_1^2 + z_1 z_2 + z_2^2 + z_1 x + z_1 + z_2) \\ \dot{z}_2 = z_2'(x)\dot{x} = 3x^2\dot{x} = 3(z_1 z_2 + z_2^2 + z_1 x + z_1 + z_2 + x^2) \end{cases} \tag{2.4}$$

Thus, we have quadratized $\dot{x}$ with z_1 and z_2 as $\dot{x}$, $\dot{z}_1$, and $\dot{z}_2$ are written as quadratic polynomials in z_1, z_2, and x. In particular, this quadratization has order 2. It can be shown using Theorem 3.1 and Proposition 3.1 that this quadratization is optimal.

3. Main Results

Our main results are Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *Suppose*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots a_2 x^2 + a_1 x + a_0$$

where $n \geqslant 5$, $a_i \in \mathbb{C}$ for all $i \in \{0, 1, 2 \dots, n-1, n\}$, and $a_n \neq 0$. A scalar polynomial ODE $\dot{x} = p(x)$ can be quadratized using exactly one new variable if and only if $p(x - \frac{a_{n-1}}{n \cdot a_n}) = a_n x^n + ax^2 + bx$ for some $a, b \in \mathbb{C}$. Moreover, this new variable can be taken to be $z := (x + \frac{a_{n-1}}{n \cdot a_n})^{n-1}$.

In Theorem 3.1, the quadratization is optimal as it has order one, and the original ODE is not quadratic.

Theorem 3.2. *Suppose*

$$\dot{x} = p_6 x^6 + p_5 x^5 + p_4 x^4 + p_3 x^3 + p_2 x^2 + p_1 x + p_0$$

for $p_i \in \mathbb{C}$ for $i \in \{0, 1, 2, 3, 4, 5, 6\}$ and $p_6 \neq 0$. Then, $\dot{x}$ can be quadratized with two new variables of the form:

$$z_1 := (\sqrt[6]{p_6} \cdot x + \frac{p_5}{6 \cdot \sqrt[6]{p_6^5}})^5 + (\frac{25p_5^3}{216\sqrt{p_6^5}} - \frac{5p_5p_4}{12\sqrt{p_6^3}} + \frac{5p_3}{8\sqrt{p_6}})(\sqrt[6]{p_6} \cdot x + \frac{p_5}{6 \cdot \sqrt[6]{p_6^5}})^2, \qquad z_2 := (\sqrt[6]{p_6} \cdot x + \frac{p_5}{6 \cdot \sqrt[6]{p_6^5}})^3.$$

Notice that $\dot{x}$ represents any degree 6 scalar polynomial ODE. For equations not satifying the requirements of Theorem 3.1, this quadratization is optimal.

Additionally, we've shown that the general form degree 6 scalar polynomial ODE cannot be quadratized with two new monomial variables, but can be quadratized with three new monomial variables (see Lemma 5.7 and Lemma 5.8).

Proposition 3.1.

(i) All degree 3 scalar polynomial ODEs can be quadratized by exactly one new variable, $z := x^2$.
(ii) All degree 4 scalar polynomial ODEs can be quadratized by exactly one new variable, $z := x^3$.
(iii) All degree 5 scalar polynomial ODEs can be quadratized by exactly two new variables, $z_1 := x^4$ and $z_2 := x^3$.

Note that in parts (i) and (ii) of Proposition 3.1, the quadratizations are optimal for precisely the same reason the quadratization in Theorem 3.1 is optimal.

4. Discussion

In our main results, we have given conditions for when one new variable is enough to quadratize a scalar polynomial ODE. In fact, this new variable has non-monomial right-hand side. We have also shown that two new non-monomial variables are enough to quadratize any degree 6 scalar polynomial ODE.

Theorem 3.1 is interesting because it demonstrates that even if we deal with high degree scalar polynomial ODEs, there is a certain form of these ODEs that can be quadratized with only one new non-monomial variable. In particular, the most interesting part of Theorem 3.1 is its use of linear shift. We consider the following scalar polynomial ODE to illustrate this important feature of Theorem 3.1:

$$\dot{x} = (x+1)^n = x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-2}x^2 + \binom{n}{n-1}x + 1 \tag{4.1}$$

In (4.1), every coefficient behind x^k for $k \in \{0, 1, 2, \ldots, n\}$ is non-zero. While it does not appear to be quadratizable by just one new variable, Theorem 3.1 tells us it is. We can say that $p(x) = (x+1)^n$ and $a_{n-1} = n$. Thus, it follows that:

$$p(x - \frac{a_{n-1}}{n}) = p(x-1) = (x-1+1)^n = x^n \tag{4.2}$$

By Theorem 3.1, since x^n is of the form $x^n + ax^2 + bx$, it can be quadratized by one new variable.

To the best of our knowledge, no algorithms consider this shift in determining the order of quadratization and thus, for arbitrarily large n, these algorithms would introduce more variables than necessary [1, 2, 6].

More precisely, we can provide a lower bound on the number of new monomial variables necessary to quadratize the scalar ODE presented in (4.1). Suppose set $S = \{z_i \mid 1 \leqslant i \leqslant k\}$ where $2 \leqslant \deg z_1 < \deg z_2 < \cdots < \deg z_k$ denotes the set of new monomial variables used to quadratize the ODE in (4.1) (we do not consider degree less than two due to Lemma 5.3). Let us also append x and 1 to the set S. Any quadratic term in our quadratization can be formed by choosing any two, not necessarily distinct, elements of set S and multiplying them together. It follows that we can form at most $\binom{k+3}{2}$ quadratic monomial terms. Since the right-hand side of (4.1) must be quadratized and contains $n+1$ monomials, we have:

$$\binom{k+3}{2} \geqslant n+1 \quad \Longrightarrow \quad k \geqslant \frac{-5+\sqrt{8n+9}}{2} \tag{4.3}$$

Here, we have provided a lower bound for k or the number of new variables introduced. For larger n, we get larger k. However, we show that simply one new variable is enough for any value of n if we consider new non-monomial variables, which is a significant improvement on monomial quadratization.

Similarly, Theorem 3.2 shows that degree 6 scalar polynomial ODEs can be quadratized by two new non-monomial variables. On the other hand, we also showed in Lemma 5.7 and Lemma 5.8 that a general degree 6 scalar polynomial ODE cannot be quadratized using two new monomial variables, but all degree 6 scalar polynomial ODEs can be quadratized using three new monomial variables. By allowing our new variables to have non-monomial right-hand side, we improve the order of the quadratization by $\sim 33\%$. Note that in the Proof of Theorem 3.2, we use a similar linear shift as described above and described in Lemma 5.1.

Altogether, our results suggest that considering quadratizations with new non-monomial variables can give us more optimal quadratizations than monomial quadratizations as we increase the degree and dimension of our ODEs. Therefore, considering non-monomial quadratization of multivariable ODEs and ODE systems may be a worthwhile pursuit. Moreover, since our results provide explicit formulas for new variables, they can be used to improve current algorithms for monomial quadratization, for example, by applying some variable shifts to the input system as suggested by Theorem 3.1.

5. Proofs and Other Results

The following three lemmas are used to prove Theorem 3.1.

Lemma 5.1. *For every scalar polynomial ODE $\dot{x} = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, there exists a unique change of variable $x \to x + \lambda$ such that a_{n-1} becomes zero.*

Proof. Let $x = y + \lambda$. Substituting for x in $\dot{x}$, we get:

$$\dot{y} = a_n(y+\lambda)^n + a_{n-1}(y+\lambda)^{n-1} + \cdots + a_2(y+\lambda)^2 + a_1(y+\lambda) + a_0$$

Binomially expanding, we have:

$$\dot{y} = a_n y^n + (a_n n\lambda + a_{n-1})y^{n-1} + o(y^{n-1})$$

Since the coefficient behind the y^{n-1} is $a_n n\lambda + a_{n-1}$, it follows that the y^{n-1} term vanishes if and only if $\lambda = \frac{-a_{n-1}}{n\cdot a_n}$. Since $n > 0$ and $a_n \neq 0$, this shift exists and is unique. ☐

Lemma 5.2. *Assume that a scalar polynomial ODE* $\dot{x} = a_n x^n + q(x)$ *with* $n \geqslant 5$ *and* $\deg q(x) \leqslant n-1$ *can be quadratized by a single new variable* $z := z(x)$. *Then,* $\deg z = n-1$.

Proof. The first term of $\dot{x}$, x^n, must be quadratized by a quadratic term in x and z. For $n \geqslant 5$, only terms z, xz, and z^2 may involve x^n, and this may happen only if $\deg z \geqslant 3$. Thus, $\deg z < \deg xz < \deg z^2$. Hence x^n must be the leading monomial of one of them. Thus, $\deg z \in \{n, n-1, n/2\}$.

If $\deg z = n$, then $\deg \dot{z} = 2n-1$. Since $\deg z^2 > 2n-1$ and the degree of any other quadratic monomial in x and z is less than $2n-1$, $\dot{z}$ cannot be quadratized. Thus, $\deg z \neq n$.

If n is odd, the only remaining option is $\deg z = n-1$, so we are done. Consider the case of even n and $\deg z = \frac{n}{2}$. Let $z = z(x) := \alpha x^{\frac{n}{2}} + r(x)$ and $\dot{x} = x^n + q(x)$ where $\deg r(x) < n/2$, $\deg q(x) < n$, and $\alpha \neq 0$. Since $\dot{z}$ must be quadratized, we have that:

$$\dot{z} = z'(x)\dot{x} = \left(\alpha \frac{n}{2} x^{\frac{n-2}{2}} + \mathrm{o}(x^{\frac{n-2}{2}})\right)(x^n + \mathrm{o}(x^n)) = \alpha \frac{n}{2} x^{\frac{3n-2}{2}} + \mathrm{o}(x^{\frac{3n-2}{2}}).$$

Since $\deg z \geqslant 3$, the degree of any quadratic polynomial in x and z is at most $2 \deg z \leqslant n$, which is less than $\frac{3n-2}{2}$ for $n \geqslant 6$. So, $\deg z \neq \frac{n}{2}$. Thus, $\deg z = n-1$. ☐

Lemma 5.3. *Suppose* $z_1, z_2, \ldots, z_m$ *quadratize some* $\dot{x} = p(x)$. *Then, the same new variables with omitted constant and linear (w.r.t.* x*) terms also quadratize this ODE.*

Proof. For each i in $\{1, 2, \ldots, m\}$, let:

$$z_i := a_i x^{k_i} + \cdots + b_i x + c_i$$

Since $z_1, z_2, \ldots, z_m$ quadratize $\dot{x}$, it follows that $\dot{x}, \dot{z}_1, \dot{z}_2, \ldots, \dot{z}_m$ are written with at most quadratic right-hand side in $x, z_1, z_2, \ldots, z_m$. For $i \in \{1, 2, \ldots, m\}$, any quadratic terms in z_i-s can be written as quadratic in $z_i - b_i x - c_i$-s and x. ☐

Proof of Theorem 3.1. We will first prove the backward direction: if $p(x - \frac{a_{n-1}}{n\cdot a_n}) = a_n x^n + ax^2 + bx$ for some $a, b \in \mathbb{C}$, then $\dot{x} = p(x)$ can be quadratized using exactly one new variable.

Suppose $\dot{x} = p(x) = a_n x^n + a_{n-1}x^{n-1} + o(x^{n-1})$ and $p(x - \frac{a_{n-1}}{n\cdot a_n}) = a_n x^n + ax^2 + bx$. So, we will shift $\dot{x}$ with the change of variables $x = y - \frac{a_{n-1}}{n\cdot a_n}$. Substituting for x in $\dot{x}$, we have that $\dot{y} = a_n y^n + ay^2 + by$ for some $a, b \in \mathbb{C}$. Let $z := y^{n-1}$. It follows that:

$$\begin{cases} \dot{y} = a_n zy + ay^2 + by \\ \dot{z} = (n-1)y^{n-2}(\dot{y}) = (n-1)(a_n y^{2n-2} + ay^n + by^{n-1}) = (n-1)(a_n z^2 + azy + bz) \end{cases}$$

Now, we will prove the forward direction: if a scalar polynomial ODE $\dot{x} = p(x)$ can be quadratized using exactly one new variable, then $p(x - \frac{a_{n-1}}{n\cdot a_n}) = a_n x^n + ax^2 + bx$ for some $a, b \in \mathbb{C}$.

Shifting x as described in Lemma 5.1, we will assume in what follows that our ODE is of the form

$$\dot{x} = a_n x^n + q(x)$$

where $\deg q(x) \leqslant n-2$ and $a_n \neq 0$. By Lemma 5.2, our new variable z must be of degree $n-1$. Thus, let:

$$z := x^{n-1} + r(x)$$

By Lemma 5.3, we can take $r(x)$ with no linear or constant term. Since $\dot{x}$ must be quadratized by z, we can write:

$$\dot{x} = a_n x^n + q(x) = a_n xz + ez + ax^2 + bx + c = a_n x^n + (a_n x + e)r(x) + ax^2 + bx + c \tag{5.1}$$

Notice that z^2 is not involved in (5.1) because for $n \geqslant 5, \deg z^2 > \deg \dot{x}$. From (5.1), it follows that:

$$q(x) = (a_n x + e)r(x) + ax^2 + bx + c \tag{5.2}$$

From (5.2), since $\deg q(x) \leqslant n-2$, we observe that $2 \leqslant d := \deg r(x) \leqslant n-3$. This implies that $e = 0$ because the ez term in (5.1) is the only term that involves x^{n-1}. However, we know that $\dot{x}$ has no x^{n-1} term.

Since $\deg r(x) \leqslant n-3$, we can write:

$$\begin{cases} r(x) = c_d x^d + c_{d-1}x^{d-1} + \ldots + c_2 x^2 \\ r'(x) = dc_d x^{d-1} + (d-1)c_{d-1}x^{d-2} + \ldots + 2c_2 x \end{cases}$$

We assume that $r(x)$ is nonzero and in fact, $2 \leqslant d \leqslant n-3$. Using the next two equations below, we will use proof by contradiction show that $r(x) = r'(x) = 0$.

We look to write $\dot{z}$ in two different ways. The first way is by direct calculation:

$$\begin{aligned} \dot{z} = z'(x)\dot{x} &= ((n-1)x^{n-2} + r'(x))(a_n x^n + a_n x r(x) + ax^2 + bx + c) \\ &= (n-1)x^{n-2} \cdot a_n x^n + (n-1)x^{n-2} \cdot a_n x r(x) + r'(x) \cdot a_n x^n + \mathrm{o}(x^{n-1+d}) \\ &= (n-1)a_n x^{2n-2} + (n-1)c_d a_n x^{n-1+d} + dc_d a_n x^{n-1+d} + \mathrm{o}(x^{n-1+d}) \end{aligned} \tag{5.3}$$

On the other hand, by definition of quadratization, the right-hand side of $\dot{z}$ must be written as at most quadratic in x and z. Thus, $\dot{z}$ must have the form:

$$\begin{aligned} \dot{z} &= (n-1)a_n z^2 + b_1 zx + b_2 z + b_3 x^2 + b_4 x + b_5 \\ &= (n-1)a_n x^{2n-2} + 2(n-1)a_n x^{n-1} r(x) + \mathrm{o}(x^{n-1+d}) \\ &= (n-1)a_n x^{2n-2} + 2(n-1)c_d a_n x^{n-1+d} + \mathrm{o}(x^{n-1+d}) \end{aligned} \tag{5.4}$$

for constants b_1, b_2, b_3, b_4, b_5.

Setting (5.3) and (5.4) equal to each other and simplifying, we obtain the following:

$$dc_d a_n x^{n-1+d} + \mathrm{o}(x^{n-1+d}) = (n-1)c_d a_n x^{n-1+d} + \mathrm{o}(x^{n-1+d}) \tag{5.5}$$

Analyzing the coefficients of the highest degree terms of each side of (5.5), we have:

$$(n-1)c_d a_n = dc_d a_n \tag{5.6}$$

Since $c_d \neq 0$ and $a_n \neq 0$, we can divide both sides of (5.6) by them:

$$n - 1 = d \tag{5.7}$$

However, since $d \leqslant n-3$, we have reached a contradiction. So, $r(x) = r'(x) = 0$.

To complete the proof, we will show that the linear term c in $\dot{x}$ must be zero. Using $r(x) = 0$, we have:

$$\dot{x} = a_n x^n + q(x), \quad q(x) = ax^2 + bx + c, \quad z := x^{n-1}$$

$$\dot{x} = a_n zx + ax^2 + bx + c$$

$$\dot{z} = z'(x)\dot{x} = (n-1)a_n x^{2n-2} + (na-a)x^n + (nb-b)x^{n-1} + (nc-c)x^{n-2} \tag{5.8}$$

Since $\dot{z}$ must be written as at most quadratic in x and z, it must be of the form:

$$\dot{z} = (n-1)a_n z^2 + b_1 zx + b_2 z + b_3 x^2 + b_4 x + b_5 \tag{5.9}$$

for some $b_1, b_2, b_3, b_4, b_5 \in \mathbb{C}$. In (5.8), we have the term $(nc-c)x^{n-2}$. However, this term cannot be quadratized for $n \geqslant 5$ because there is no term with degree $n-2$ in (5.9). Thus, we find that $nc - c = 0$. Since $n \geq 5$, $c = 0$. All other terms in (5.8) can be written using some quadratic combination of z and x. ☐

The following two lemmas and corollary are used to prove Theorem 3.2.

Lemma 5.4. *Suppose $z_1, z_2, \ldots, z_k$ quadratize $\dot{x}$. Consider z_1 and z_2. Let $a, b \in \mathbb{C}$. If $z_1, z_2, \ldots, z_k$ quadratize $\dot{x}$, then $az_1 + bz_2, z_2, \ldots, z_k$ also quadratize $\dot{x}$.*

Proof. Since $z_1, z_2, \ldots, z_k$ quadratize $\dot{x}$, it follows that $\dot{x}, \dot{z}_1, \dot{z}_2, \ldots, \dot{z}_k$ are written with at most quadratic right-hand side in $x, z_1, z_2, \ldots, z_k$. Any quadratic term in z_1, z_2, $\ldots$, z_k, and x can be written as quadratic in $az_1 + bz_2$, z_2, z_3, $\ldots$, z_k, and x. Thus, it holds that if $z_1, z_2, \ldots, z_k$ quadratize $\dot{x}$, then $az_1 + bz_2, z_2, \ldots, z_k$ also quadratizes $\dot{x}$. □

Corollary 5.1. Suppose $z_1, z_2, \ldots z_k$ quadratize $\dot{x}$ where each term has leading coefficient c_i. Then, each of the new variables can have a distinct degree.

Proof. The proof follows directly from Lemma 5.4. □

Lemma 5.5. *Let $\dot{x} = p(x)$ be a scalar polynomial ODE with $\deg p := n \geqslant 5$ that can be quadratized with $k \geqslant 2$ new variables. Let $z_1, z_2, z_3, \ldots, z_k$ denote the k new variables used to quadratize $\dot{x}$. Suppose that for any $i \in \{1, 2, 3, \ldots, k\}$, $\deg z_i \leqslant n-1$. For $S = \{\deg z_1, \deg z_2, \deg z_3, \ldots, \deg z_k\}$, it holds that $n - 1 \in S$.*

Proof. Assume for contradiction that $n - 1 \notin S$. Thus, it follows that $\max(S) \leqslant n-2$. Assume that $\max(S)$ is r for some $2 \leqslant r \leqslant n-2$. Let z_r be the new variable with degree r. It follows that $\deg z_r = n + r - 1$. Notice that the largest degree we can form with our new variables is $2r$. Thus, it follows that $2r \geqslant n + r - 1$. However, this implies that $r \geqslant n - 1$. Thus, $n - 1 \in S$. □

Lemma 5.6. *Assume that $\dot{x} = p(x)$ be a scalar polynomial ODE with $\deg p := n \geqslant 5$ can be quadratized by two new variables. Then, it can be quadratized using two new variables y and z, one of which has degree $n-1$.*

Proof. Let y and z be some quadratizing variables. By Corollary 5.1, they can be assumed to have distinct degrees. By Lemma 5.3, y and z can be taken with no linear or constant term. So, we will assume $\deg y > \deg z \geqslant 2$. It holds that $\deg \dot{y} = n - 1 + \deg y$.

We will first consider the case where $\deg y \leqslant n-1$. The proof of this follows directly from Lemma 5.5.

Now, we will consider the case where $\deg y \geqslant n$. If $\deg y \geqslant n$, then y^2 cannot appear in the right-hand side of $\dot{y}$ because $\deg y^2 > \deg \dot{y}$. We will look to write the leading term of $\dot{y}$ as the highest degree term of some linear combination of the quadratic terms in x, y, and z. The leading term of $\dot{y}$ cannot be written as the highest degree term of any linear combination of $1, x, x^2, y, z, yx$, or zx because their degrees are too small.

Thus, we are left with z^2 and yz. Note that $\deg z^2 < \deg yz$. Thus, one of them must have degree $n - 1 + \deg y$. If $\deg yz = n - 1 + \deg y$, then it holds that $\deg z = n - 1$.

To finish the proof, we will show that z^2 cannot have degree $n - 1 + \deg y$. If $\deg z^2 = n - 1 + \deg y$, then $\deg z \geqslant n$ and $\deg y \geqslant n + 1$. Additionally, $\deg z^2 = n - 1 + \deg y$ implies that the right-hand side must be even. So, $\deg y = n - 1 + 2k$ for some $k \in \mathbb{N}$. It follows that $\deg z^2 = 2n - 2 + 2k$, $\deg z = n - 1 + k$, and $\deg \dot{z} = 2n - 2 + k$. We must have that the leading term of $\dot{z}$ can be written as the highest degree term of some linear combination of the quadratic terms in x, y, and z. No linear combination of $1, x, x^2, y, z, yx$, and zx have degree $2n - 2 + k$ because the degree of each of these terms is too small. Furthermore, no linear combination of z^2, y^2, and yz – each of which have distinct degrees – has degree $2n - 2 + k$ because the degree of each of these terms is too large. Thus, z^2 cannot have degree $n - 1 + \deg y$. □

Proof of Theorem 3.2. Suppose

$$\dot{x} = p_6 x^6 + p_5 x^5 + p_4 x^4 + p_3 x^3 + p_2 x^2 + p_1 x + p_0$$

Let $x = \frac{y}{\sqrt[6]{p_6}} - \frac{p_5}{6 \cdot p_6}$. This change of variable aims to make p_5 equal to zero as demonstrated by Lemma 5.1 and make the leading coefficient 1. Substituting for x, we have:

$$\dot{x} = y^6 + \Big(\frac{p_4}{\sqrt[3]{p_6^2}} - \frac{5p_5^2}{12\sqrt[3]{p_6^5}}\Big)y^4 + \Big(\frac{5p_5^3}{27\sqrt{p_6^5}} + \frac{p_3}{\sqrt{p_6}} - \frac{2p_5p_4}{3\sqrt{p_6^3}}\Big)y^3 + \Big(\frac{p_5^2p_4}{6\sqrt[3]{p_6^7}} + \frac{p_2}{\sqrt[3]{p_6}} - \frac{p_5p_3}{2\sqrt[3]{p_6^4}} - \frac{5p_5^4}{144\sqrt[3]{p_6^{10}}}\Big)y^2$$
$$+ \Big(\frac{p_5^5}{324\sqrt[6]{p_6^{25}}} + \frac{p_5^2p_3}{12\sqrt[6]{p_6^{13}}} - \frac{p_5^3p_4}{54\sqrt[6]{p_6^{19}}} - \frac{p_5p_2}{3\sqrt[6]{p_6^7}} + \frac{p_1}{\sqrt[6]{p_6}}\Big)x + \Big(\frac{p_5^4p_4}{1296p_6^4} - \frac{5p_5^6}{46656p_6^5} - \frac{p_5^3p_3}{216p_6^3} - \frac{p_5p_1}{1296p_6^4} + p_0\Big)$$

For simplicity of notation, we will write:

$$\dot{x} = q_0 + q_1 y + q_2 y^2 + q_3 y^3 + q_4 y^4 + y^6$$

Notice that $\frac{5q_3}{8} = \frac{25p_5^3}{216\sqrt{p_6^5}} + \frac{5p_3}{8\sqrt{p_6}} - \frac{5p_5p_4}{12\sqrt{p_6^3}}$. By Lemma 5.6, one of the new variables has degree 5. Thus, let:

$$z_1 := y^5 + \frac{5q_3}{8}y^2, \qquad z_2 := y^3$$

It follows that for any constants $c_1, c_2, c_3 \in \mathbb{C}$, we have that:

$$(5.10) \quad \begin{cases} \dot{x} = (1-c_1)z_1y + c_1z_2^2 + (\frac{5c_1q_3+3q_3}{8})z_2 + q_4z_2y + q_2y^2 + q_1y + q_0 \\ \dot{z}_1 = 5z_1^2 + (5q_1 - \frac{15q_3q_4}{8})z_1 + (5q_2 - c_2)z_1y + (\frac{5c_2q_3 - 15q_2q_3}{8})z_2 + (5q_0 - \frac{45q_3^2}{64})z_2y + c_2z_2^2 \\ \qquad + 5q_4z_1z_2 + (\frac{75q_3^2q_4 - 120q_1q_3}{64})y^2 + \frac{5q_0q_3}{4}y \\ \dot{z}_2 = 3z_1z_2 + \frac{9q_3}{8}z_1 + (3q_4 - c_3)z_1y + (3q_1 + \frac{5c_3q_3 - 15q_3q_4}{8})z_2 + 3q_2z_2y + c_3z_2^2 + (3q_0 - \frac{45q_3^2}{64})y^2 \end{cases}$$

□

Notice that the presence of constants c_1, c_2, c_3 in Theorem 3.2 suggests that there exists an infinite number of possible quadratizations of a degree 6 scalar polynomial ODE with two new variables.

Lemma 5.7. *There exists scalar polynomial ODEs of degree* 6 *which cannot be quadratized using two new monomial variables.*

Proof. Consider

$$\dot{x} = p_6x^6 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$$

where $p_6, p_4, p_3, p_2, p_1, p_0 \in \mathbb{C}\backslash\{0\}$. We use this form for $\dot{x}$ to reflect the possible use of shift on a general form degree 6 scalar polynomial ODE as discussed in Lemma 5.1. By Lemma 5.6, one of the new variables must have degree 5. Thus, for monomial quadratization, we have the following cases:

$$\begin{cases} Case\ 1: z_1 := x^5, z_2 := x^2 \\ Case\ 2: z_1 := x^5, z_2 := x^3 \\ Case\ 3: z_1 := x^5, z_2 := x^4 \\ Case\ 4: z_1 := x^5, z_2 := x^{5+k} \text{ for } k \in \mathbb{N} \end{cases}$$

Notice that in Cases $1, 2$, and 3, $\max\{\deg \dot{x}, \deg \dot{z}_1, \deg \dot{z}_2\} = 10$. In Case 1, x^8 in $\dot{z}_1$ cannot be written as quadratic in x, z_1, z_2. In Case 2 and Case 3, x^7 in $\dot{z}_1$ cannot be written as quadratic in x, z_1, and z_2. In Case 4, x^3 and x^4 in $\dot{x}$ cannot be written as quadratic in x, z_1, and z_2. Thus, 2 new monomial variables are not enough to quadratize all degree 6 scalar polynomial ODEs. □

Lemma 5.8. *All degree* 6 *scalar polynomial ODEs can be quadratized by three new monomial variables,* $z_1 := x^5, z_2 := x^4, z_3 := x^3$.

Proof. Let

$$\dot{x} = p_6x^6 + p_5x^5 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$$

with $p_6 \neq 0$. It follows that for the constants c_i:

$$(5.11)\qquad \begin{cases} \dot{x} = p_6 z_1 x + p_5 z_1 + p_4 z_2 + p_3 z_3 + p_2 x^2 + p_1 x + p_0 \\ \dot{z}_1 = 5x^4 \dot{x} = 5p_6 z_1^2 + 5p_5 z_1 z_2 + 5p_4 c_1 z_2^2 + 5p_4(1-c_1) z_1 z_3 + 5p_3 z_2 z_3 + 5p_2 c_2 z_1 x \\ \qquad +5p_2(1-c_2) z_3^2 + 5p_1 c_3 z_1 + 5p_1(1-c_3) z_2 x + 5p_0 c_4 z_2 + 5p_0(1-c_4) z_3 x \\ \dot{z}_2 = 4x^3 \dot{x} = 4p_6 z_1 z_2 + 4p_5 c_5 z_2^2 + 4p_5(1-c_5) z_1 z_3 + 4p_4 z_2 z_3 + 4p_3 c_6 z_1 x + 4p_3(1-c_6) z_2^2 \\ \qquad +4p_2 c_7 z_1 + 4p_2(1-c_7) z_2 x + 4p_1 c_8 z_2 + 4p_1(1-c_8) z_3 x + 4p_0 z_3 \\ \dot{z}_3 = 3x^2 \dot{x} = 3p_6 c_9 z_2^2 + 3p_6(1-c_9) z_1 z_3 + 3p_5 z_2 z_3 + 3p_4 c_{10} z_1 x + 3p_4(1-c_{10}) z_3^2 \\ \qquad +3p_3 c_{11} z_1 + 3p_3(1-c_{11}) z_2 x + 3p_2 c_{12} z_2 + 3p_2(1-c_{12}) z_3 x + 3p_1 z_3 + 3p_0 x^2 \end{cases}$$

□

Proof of Proposition 3.1. Part (i). All degree 3 scalar polynomial ODEs can be quadratized by exactly one new variable, $z := x^2$.

Let $\dot{x} = p_3 x^3 + p_2 x^2 + p_1 x + p_0$ where $p_i \in \mathbb{C}$ for $i = 0,1,2,3$ and $z := x^2$. It follows that:

$$(5.12)\qquad \begin{cases} \dot{x} = p_3 z x + p_2 x^2 + p_1 x + p_0 \\ \dot{z} = 2x\dot{x} = 2p_3 z^2 + 2p_2 z x + 2p_1 x^2 + 2p_0 x \end{cases}$$

Part (ii). All degree 4 scalar polynomial ODEs can be quadratized by exactly one new variable, $z := x^3$.

Let $\dot{x} = q_4 x^4 + q_3 x^3 + q_2 x^2 + q_1 x + q_0$ where each $q_i \in \mathbb{C}$. By Lemma 5.1, any degree 4 scalar polynomial ODE can be uniquely shifted such that the coefficient behind x^3 becomes zero. Applying the change of variables $x = y - \frac{q_3}{4 \cdot q_4}$, we get that $\dot{y} = q_4 y^4 + p_2 y^2 + p_1 y + p_0$ for some p_j-s $\in \mathbb{C}$. Let $z := y^3$. It follows that:

$$(5.13)\qquad \begin{cases} \dot{y} = q_4 z y + p_2 y^2 + p_1 y + p_0 \\ \dot{z} = 3y^2 \dot{y} = 3q_4 z^2 + 3p_2 z y + 3p_1 z + 3p_0 y^2 \end{cases}$$

Part (iii). All degree 5 scalar polynomial ODEs can be quadratized by exactly two new variables, $z_1 := x^4$ and $z_2 := x^3$.

Let $\dot{x} = p_5 x^5 + p_4 x^4 + p_3 x^3 + p_2 x^2 + p_1 x + p_0$ where $p_i \in \mathbb{C}$ for $i = 0,1,2,3,4,5$ and $p_5 \neq 0$. Also, let $z_1 := x^4$ and $z_2 := x^3$. It follows that:

$$(5.14)\qquad \begin{cases} \dot{x} = p_5 z_1 x + p_4 z_1 + p_3 z_2 + p_2 x^2 + p_1 x + p_0 \\ \dot{z}_1 = (z_1')(\dot{x}) = 4x^3(\dot{x}) = 4p_5 z_1^2 + 4p_4 z_1 z_2 + 4p_3 z_2^2 + 4p_2 z_1 x + 4p_1 z_1 + 4p_0 z_2 \\ \dot{z}_2 = (z_2')(\dot{x}) = 3x^2(\dot{x}) = 3p_5 z_1 z_2 + 3p_4 z_2^2 + 3p_3 z_1 x + 3p_2 z_1 + 3p_1 z_2 + 3p_0 x^2 \end{cases}$$

□

6. Computational Techniques

In this section, we describe the computational techniques used in order to gain the intuition for Theorem 3.1 and find the form of the quadratization presented in Theorem 3.2. The main tool used for our computation was Gröbner bases.

A Gröbner basis is a set of multivariate polynomials that has desirable algorithmic properties. It holds that every set of polynomials can be transformed into a Gröbner basis. Gröbner basis computation is an effective way of reducing or solving systems of equations and generalizes Gaussian elimination and the Euclidean algorithm for polynomials. For more, see [8, 9].

6.1. Quadratization with One New Variable. In this subsection, we will outline the computational experiments used to gain the intuition for Theorem 3.1. For simplicity, we will focus on degree 5 scalar polynomial ODE since it is the smallest degree for which Theorem 3.1 can be applied.

Using Lemma 5.1, Lemma 5.2, and Lemma 5.3, we introduce the following set-up:

$$\begin{cases} \dot{x} = p_5x^5 + p_3x^3 + p_2x^2 + p_1x + p_0 \\ z := x^4 + q_3x^3 + q_2x^2 \end{cases}$$

We define our polynomial ring $R = \mathbb{C}[p_0, p_1, p_2, p_3, p_4, q_3, q_2]$.

We ask the following elimination question: for what values of p_i does there exist values of q_j such that $\dot{x}$ can be written as some linear combination of $S_1 = \{1, x, x^2, z, zx\}$ and $\dot{z}$ can be written as some linear combination of $S_2 = \{1, x, x^2, z, zx, z^2\}$? Notice that z^2 is not in S_1 because $\deg z^2 = 8$, but $\deg \dot{x} = 5$.

In order to answer our question, we produce the following two matrices where each entry is defined by the coefficient behind the term that corresponds to the row in the function that corresponds to the column:

$$\dot{x}\text{ matrix} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x & x^2 & z & xz & \dot{x} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & p_0 \\ 0 & 1 & 0 & 0 & 0 & p_1 \\ 0 & 0 & 1 & q_2 & 0 & p_2 \\ 0 & 0 & 0 & q_3 & q_2 & p_3 \\ 0 & 0 & 0 & 1 & q_3 & p_4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\dot{z}\text{ matrix} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & x & x^2 & z & xz & z^2 & \dot{z} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3 & q_2 & 0 & 2p_2q_2 + 3p_1q_3 + 4p_0 \\ 0 & 0 & 0 & 1 & q_3 & q_2^2 & 2p_3q_2 + 3p_2q_3 + 4p_1 \\ 0 & 0 & 0 & 0 & 1 & 2q_2q_3 & 2p_4q_2 + 3p_3q_3 + 4p_2 \\ 0 & 0 & 0 & 0 & 0 & q_3^2 + 2q_2 & 4p_3 + 3p_4q_3 + 2q_2 \\ 0 & 0 & 0 & 0 & 0 & 2q_3 & 3q_3 + 4p_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \end{array}$$

Notice that in the $\dot{x}$ matrix, the first 5 column vectors are linearly independent. Thus, $\dot{x}$ can be written as some linear combination of $S_1 = \{1, x, x^2, z, zx\}$ iff all 6 column vectors in the $\dot{x}$ matrix are linearly dependent. This happens precisely when the determinant of the $\dot{x}$ matrix equals 0.

Similarly, notice that in the $\dot{z}$ matrix, the first 6 column vectors are linearly independent. Thus, $\dot{z}$ can be written as some linear combination of $S_2 = \{1, x, x^2, z, zx, z^2\}$ iff all 7 column vectors in the $\dot{z}$ matrix are linearly dependent. Since the $\dot{z}$ matrix in non-square, this happens precisely when the minors of the $\dot{z}$ matrix equal 0.

Thus, we define our set of polynomials as the determinant of the $\dot{x}$ matrix and the minors of the $\dot{z}$ matrix in p_i-s and q_j-s. More precisely, the problem we look to solve is: for what values of p_i-s do there exist q_j-s such that the determinant of the $\dot{x}$ matrix and the minors of the $\dot{z}$ matrix equal zero. The way we solve this problem is by computing the Gröbner basis of this set of polynomials and then, selecting only the expressions in p_i-s. While this does not give the complete set of p_i-s, it gives us the closure of this set [3, Theorems 2 and 3, §3.1]. This was enough to give us the intuition for the proof of Theorem 3.1.

The polynomials only in p_i-s of the computed Gröbner basis are:

$$\begin{cases} p_5p_0 = 0 \\ p_3 = 0 \end{cases}$$

Since $p_5 \neq 0$, it follows that $p_0 = p_3 = 0$. Testing with higher degree scalar polynomial ODEs, the same pattern holds, giving us the intuition for the proof of Theorem 3.1.

6.2. Quadratization with Two New Variables. In this subsection, we will focus on the methods we used to find the quadratization presented in Theorem 3.2. Our goal was to find the full characterization of the quadratization of a scalar polynomial ODE of degree 6 with exactly two new variables or determine that it is not possible.

By Lemma 5.6, we know one of the new variables has degree 5. Using Corollary 5.1, we can say that our second new variable does not have degree 5. To start, we assume that the other new variable has degree less than 5. So, we try every possible combination of degrees of variables. In other words, we try two new variables of degree 5 and degree 4, degree 5 and degree 3, and degree 5 and degree 2. By Lemma 5.3, we do not consider degree 1 and degree 0 for our second new variable. Here, we will simply outline the correct solution where our first new variable has degree 5 and our second new variable has degree 3. However, in order to thoroughly experiment with degree 6 scalar polynomial ODEs, we conducted this computation with other degrees for our second new variable. We introduce the following setup.

Using Lemma 5.1, let:

$$\dot{x} = x^6 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$$

Using Lemma 5.3, we can take our new variables with no linear or constant term. Let:

$$z_1 := x^5 + q_4x^4 + q_2x^2, \qquad z_2 := x^3 + r_2x^2$$

Note that we used Lemma 5.4 to take z_1 with no x^3 term.

We would like to know when $\dot{x}$, $\dot{z}_1$, and $\dot{z}_2$ can be written as at most quadratic combination of x, z_1, z_2. In other words, we would like to know when $\dot{x}$, $\dot{z}_1$, and $\dot{z}_2$ can be written as a linear combination of the terms in $S = \{1, x, x^2, z_1, z_1x, z_1^2, z_2, z_2x, z_2^2, z_1z_2\}$. Thus, this happens when:

$$\text{(6.1)} \qquad \begin{cases} \dot{x} - c_1 - c_2x - c_3x^2 - c_4z_1 - c_5z_1x - c_6z_2 - c_7z_2x - c_8z_2^2 = 0 \\ \dot{z}_1 - c_9x - c_{10}x^2 - c_{11}z_1 - c_{12}z_1x - c_{13}z_1^2 - c_{14}z_2 - c_{15}z_2x - c_{16}z_2^2 - c_{17}z_1z_2 = 0 \\ \dot{z}_2 - c_{18}x - c_{19}x^2 - c_{20}z_1 - c_{21}z_1x - c_{22}z_2 - c_{23}z_2x - c_{24}z_2^2 - c_{25}z_1z_2 = 0 \end{cases}$$

Note that $\dot{z}_1$ and $\dot{z}_2$ do not have constant terms by definition. Thus, we define our polynomial ring as $R = \mathbb{C}[p_4, p_3, p_2, p_1, p_0, q_2, r_2, \bar{c}]$ where $\bar{c} = [c_1, c_2, \ldots, c_{25}]$. Our system of equations is the coefficients behind each monomial term x^i in each equation in (6.1) set to zero (this system of equations is referred to as *polys* in our code). Precisely, we ask the following question: for what values of p_i-s do there exist c_j-s, q_k-s, and r_l-s that satisfy (6.1).

In order to reduce the complexity of our computation, we computed the coefficients behind each x^i term of each equation of (6.1). These terms form the system of equations for which we would like to find a Gröbner basis. Taking these coefficients equal to zero, we aimed to replace as many c_j-s as possible in (6.1) with terms in p_4, p_3, p_2, p_1, p_0 in order to reduce the number of terms in our polynomial ring. For example, the coefficient behind the linear term in the second equation is $c_9 - 2q_2p_0$. This term is in our Gröbner basis. Since the left-hand side of each equation in (6.1) must equal zero, we have that $c_9 - 2q_2p_0 = 0$. This gives us that $c_9 = 2q_2p_0$. Thus, we replace c_9 in our system of equations with $2q_2p_0$ and remove c_9 from our polynomial ring. Extending this method to include any terms in our system of equations and its ideal, we simplify our computation by reducing the number of variables we must work with. We provide all of the exact variable replacements in the order we replaced them in the following table (the bold horizontal line denotes the place where we reloaded the worksheet and analyzed the new outputs to make the rest of the replacements):

Variable Replacement List	
Original	**Replacement**
c_1	p_0
c_2	p_1
c_5	$1 - c_8$
c_9	$2q_2q_0$
c_{13}	5
c_{25}	3
q_4	0
r_2	0
q_2	$\frac{5}{8}p_3$
c_3	p_2
c_4	0
c_6	$\frac{5}{8}c_8p_3\frac{3}{8}p_3$
c_7	p_4
c_{10}	$\frac{-15}{8}p_1p_3 + \frac{75}{64}p_3^2p_4$
c_{11}	$5p_1 - \frac{15}{8}p_3p_4$
c_{12}	$5p_2 - c_{16}$
c_{14}	$\frac{5}{8}c_{16}p_3 - \frac{15}{8}p_2p_3$
c_{15}	$5p_1 - \frac{45}{64}p_3^2$
c_{17}	$5p_4$
c_{19}	$3p_0 - \frac{45}{64}p_3^2$
c_{20}	$\frac{9}{8}p_3$
c_{21}	$3p_4 - c_{24}$
c_{22}	$\frac{5}{8}c_{24}p_3 + 3p_1 - \frac{15}{8}p_3p_4$
c_{23}	$3p_2$

After making these replacements, we filter through the Gröbner basis and select terms only in p_i-s. We obtain that the Gröbner basis of the intersection of our ideal and the polynomial ring of only p_i-s is the empty set. This suggests that for any p_i-s, there exist c_j-s, q_k-s, and r_l-s that satisfy (6.1). This confirms that all degree 6 scalar polynomial ODEs can be quadratized with exactly two new variables. Observing the replacements we made, we find the form of our quadratization.

7. Conclusion

We have shown in Theorem 3.1 that a scalar polynomial ODE $\dot{x} = p(x) = a_nx^n + a_{n-1}x^{n-1} + o(x^{n-1})$ with $n \geqslant 5$ can be quadratized using exactly one new variable if and only if $p(x - \frac{a_{n-1}}{n \cdot a_n}) = a_nx^n + ax^2 + bx$ for some $a, b \in \mathbb{C}$. We have also shown in Theorem 3.2 that all degree 6 scalar polynomial ODEs can be quadratized with two new non-monomial variables. Finally, we have shown that all degree 3 and 4 scalar polynomial ODEs can be quadratized with one new monomial variable and all degree 5 scalar polynomial ODEs can be quadratized with two new monomial variables. These results indicate that adding non-monomial variables may lead to substantially more optimal quadratization than the monomial ones used in the current software. They also give basic intuition about how to exploit non-monomiality (e.g., via shifts as in Theorem 3.1).

We employed computational techniques that made use of Gröbner Bases to help us gain intuition for Theorem 3.1 and find the form of the new variables and quadratization in Theorem 3.2. Our code is attached as a separate file.

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