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A Nonparametric Swiss Army Knife for Medicine

By Matthew R. Francis

The complexity of living things is frequently humbling for mathematicians. Even a single cell contains a plethora of processes and complicated interactions that tractable mathematical models cannot easily describe. Researchers have applied nonlinear dynamics, mechanical analogs, and numerous other techniques to understand biological systems, but the tradeoffs of modeling often err on the side of reductionism.

For this reason, Heather Harrington of the University of Oxford and her collaborators are turning to global mathematical methods and drawing on experimental data to identify the best techniques. Harrington described several of these methods during her invited talk at the 2021 SIAM Conference on Applications of Dynamical Systems,[1] which took place virtually earlier this year.

"The way that we look at dynamical systems is usually in a small region of the parameter space," Harrington said. This approach is helpful if one knows a lot about the model and its parameters, but it can be hard to extract detailed predictions from the model if the parameters in question range over large values. "In biology, we often don't know if the system is very close to a value in parameter space because the variables or parameters are difficult to measure or the data is too messy," she added.

In particular, parameters may vary significantly between healthy and diseased systems — or even between two healthy states within the same system. "Sometimes you can look at this as changes in parameters with bifurcation theory," Harrington said. "But if you want to look at what's possible in the system more globally, studying the structure of the equations algebraically can be helpful."

She and her colleagues have developed nonparametric techniques that use algebraic geometry, topology, and other mathematical fields to study problems in biology and medicine. They apply whichever mathematical methods are most enlightening to examine signaling in both healthy and cancerous cells, as well as blood vessel growth in tumors.

"What I'm looking at really is application driven," Harrington said. "If the data that we have is spatio-temporal, then the models I want to look at are too. It's really about finding the appropriate math for the problem."

She Wnt Thataway

Textbook illustrations of cells generally make them look simple, with spheres and bean-like shapes that sit peacefully in jelly. But real cells are phenomenally complicated; organelles move around, proteins and RNA carry instructions to the various parts, and membranes allow some chemicals to pass through while repelling others. All of these processes maintain a cell's functions, manage when it divides, and determine when it dies.

One such set of processes is the Wnt signaling pathway,[2] which passes signals through the membranes that govern actions like cell growth and movement. These chemical pathways are important for embryonic development and are disrupted in cancer. They also evolved early in the timeline

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[1] https://www.siam.org/conferences/cm/conference/ds21

[2] "Wnt" is pronounced "wint" and stands for "wingless-related integration site" because of the way that it controls wing growth in fruit flies.

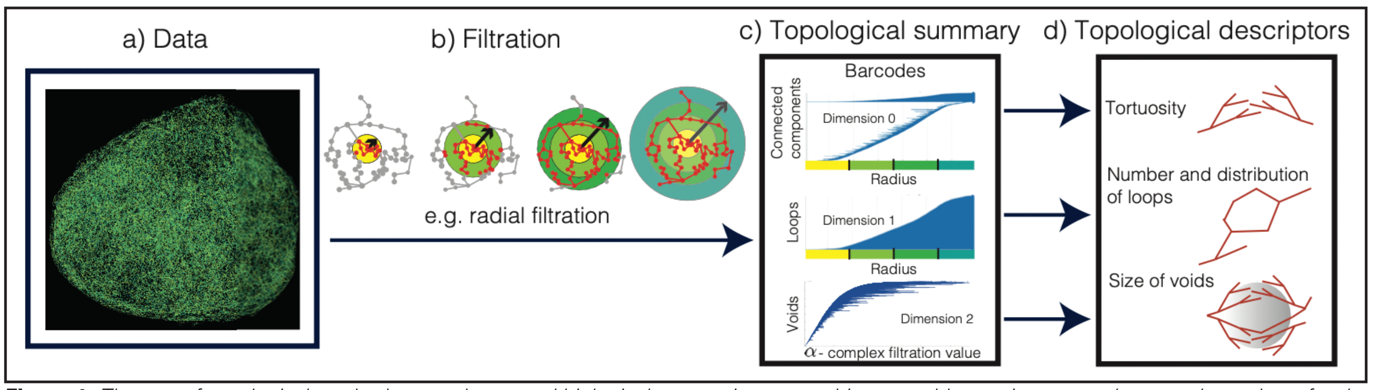

Figure 1. *The use of topological methods to analyze a real biological system into something tractable requires several steps, shown here for the growth of blood vessels in a tumor. Figure courtesy of [3].*

Removing Kalman from Ensemble Kalman Filtering

By Paul Davis

Jeff Anderson builds forecast systems for the U.S. National Center for Atmospheric Research in Boulder, Co. He began his invited talk at the 2021 SIAM Conference on Computational Science and Engineering,[1] which took place virtually earlier this year, with a potentially unsettling threat: he would remove the "Kalman" from ensemble Kalman filtering.

Anderson's objective was hardly a personal vendetta. A forecast system assimilates weather forecasts from models and weather observations from instruments, uses the observations to reduce inevitable errors in the predicted state, and feeds this improved state vector to the prediction model as the initial condition for its next run. In precise mathematical terms, the Kalman filter is the best tool for predicting a system's state (and the probability density function of that state) from noisy observations of its past, given linear process models and additive Gaussian noise in processes and measurements, among other conditions.

When all of these particular conditions are fulfilled, Kalman can compare the system model's prediction of what *should* happen with the system's observations of what *did* happen — fully exploiting the requirement that *should* and *did* are subject to Gaussian error. This comparison can filter the noise that disrupts a model's predictions and the errors that corrupt the associated measurements to produce the most accurate estimate of a system's present state and its corresponding covariance matrix.

If Kalman is mathematically the best possible filter for the heart of a weather forecasting system, why should Anderson want it removed? Simply put, the hundreds of millions of state variables in large Earth system models induce computational congestive heart failure. The mathematical derivation of the classic Kalman filter is a relatively straightforward exercise that involves inverses of covariance matrices. Computationally, inverting matrices means solving linear systems — a task whose workload grows faster than the square of the number of variables.

Rather than excising Kalman by radical surgery, Anderson characterized his approach as "an evolutionary attempt to eliminate the assumptions" that were made when the Kalman filter was developed. The evolutionary steps that he outlined amount to the one-by-one removal of some formerly essential requirements: a linear model to advance the covariance estimate, a linear forward operator, an unbiased estimate of covariance, an unbiased model prior, and (almost) a Gaussian prior.

Setting the stage for his reductive strategy, Anderson offered a graphic depiction of the filtering challenge (see Figure 1). The two-dimensional state variable is in the horizontal plane. The prior distribution in green signifies the outcome of the previous predictive step, and the likelihood in red captures what the instrument makers know about the uncertainty in their observations.

Mathematically speaking, the posterior in blue is the product of the prior with the likelihood, which is normalized by the denominator of the expression in Figure 1. Computationally, one calculates the statistics of the posterior state estimate—the shape of the blue surface—by solving large linear systems that are built from covariance matrices. On a geophysical scale, the dual burdens of storage and computational cost are prohibitive.

See **Kalman Filtering** on page 2

[1] https://www.siam.org/conferences/cm/conference/cse21

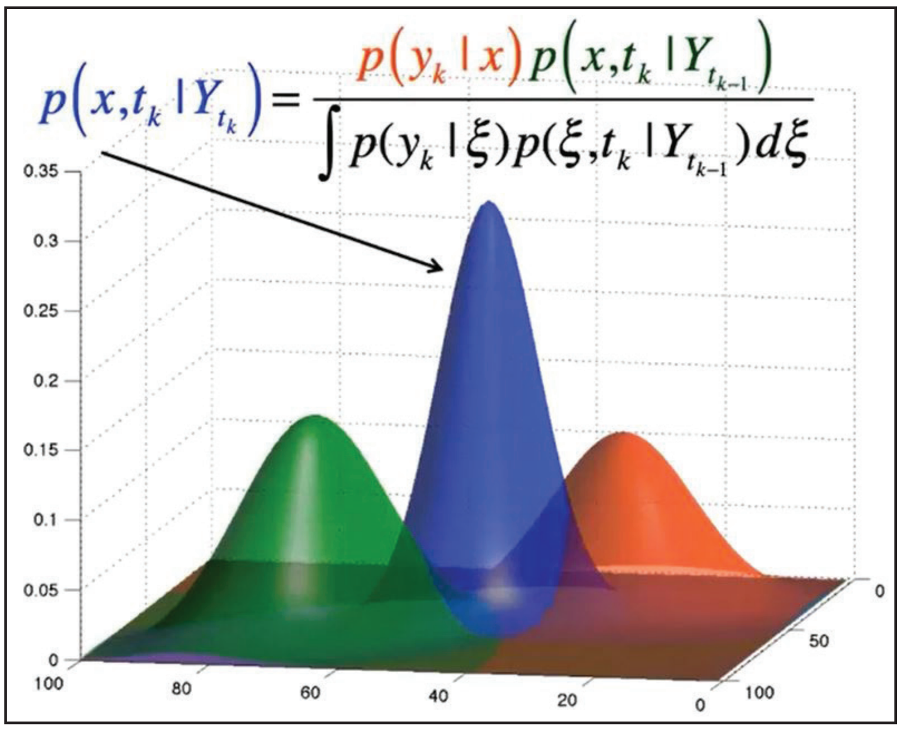

Figure 1. *Filtering a two-dimensional state variable in the horizontal plane. The prior distribution is in green and the measurement error likelihood is in red. The posterior in blue is the product of the prior with the likelihood, normalized by the denominator. Figure courtesy of Jeff Anderson.*

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Kalman Filtering

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To avoid the expense of solving Kalman's enormous linear systems, Anderson introduced an ensemble of smaller elements that ultimately capture the original system's critical features. Imagine the prior state distribution (the green component in Figure 1, on page 1) as a surface that was fit to an ensemble of prior states — a set of well-selected points in the horizontal plane. Next, advance each member of this prior ensemble a single time step to a posterior with the same mean and covariance as the blue posterior surface in Figure 1.

From a practical standpoint, Anderson and his colleagues have shown that constructing and advancing such an ensemble is entirely feasible. A deterministic update of a single observed variable "can compute the impact of an observation on each state variable independently," he said. Anderson used the measurement error likelihood to shift the observed variable, then calculated the corresponding shift in the weakly correlated unobserved state variable. The process amounts to regressing shifts in the observed variable onto shifts in the unobserved variable.

This process repeats when all ensemble members have been updated. Because the regression of observation increments onto state variable increments can be performed independently, computations can be distributed across the nodes of a parallel machine to attain even faster analyses.

Anderson approaches ensemble filtering with the realization that "making the core problem simpler allows you to introduce added complexity" at a later time, he said. Certainly, this ensemble formulation avoids both massive matrix inversions and the requirement of linear relations between observations and states.

Anderson presented two computational data assimilation experiments that are examples of such "added complexity." Both utilize the Lorenz 96 40-variable system: $dX_i/dt = (X_i + 1 - X_i - 2)$ $X_i - 1 - X_1 + F$. It behaves a bit like models of the weather within bands of latitude around the Earth. One experiment explored data localization (biased covariance) and the other dealt with model errors (biased model prior).

Data like temperature observations in Denver, Co., is obviously more important to temperature prediction in Boulder than an observation from Antarctica. Anderson used simulations with the Lorenz 96 40-variable system to illustrate the success of localizing observations. Figure 2a depicts an early stage of a set of these simulations; the red curve is the "truth" and each green asterisk is an observed value that is collected at its location along the horizontal axis. Figure 2b displays the local weighting of three of these observations, and Figure 3 demonstrates the power of such a localization after a few hundred more iterations. The blue localized ensemble predictions in Figure 3a hew closely to the red "truth," while the blue unweighted predictions in 3b deviate wildly.

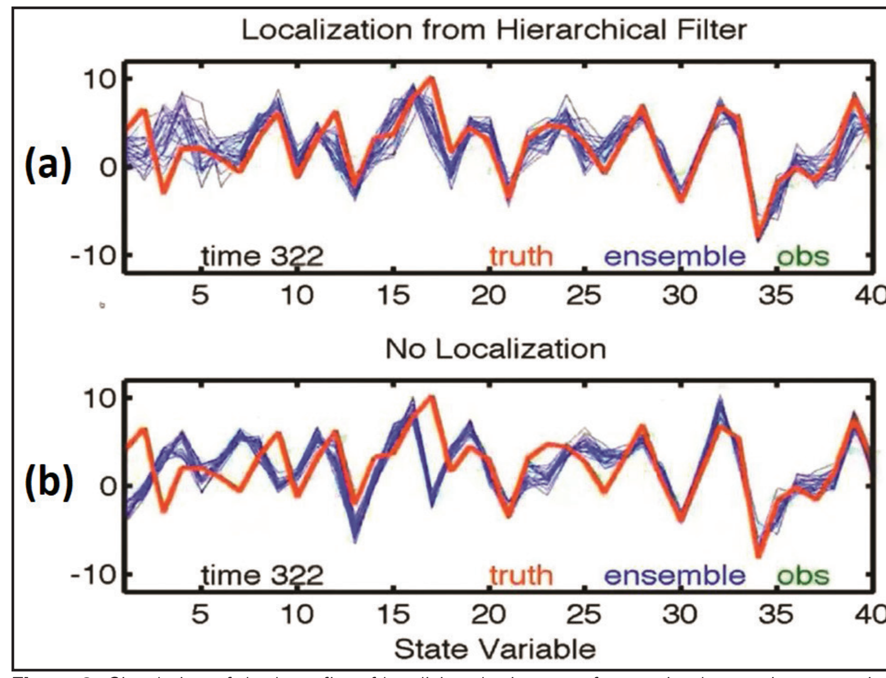


Figure 3. *Simulation of the benefits of localizing the impact of errors in observations at various locations on the horizontal axis in the Lorentz 96 system.* **3a.** *The blue ensemble simulations with error localization closely track the red "truth."* **3b.** *Simulations without localization deviate wildly. Figure courtesy of Jeff Anderson.*

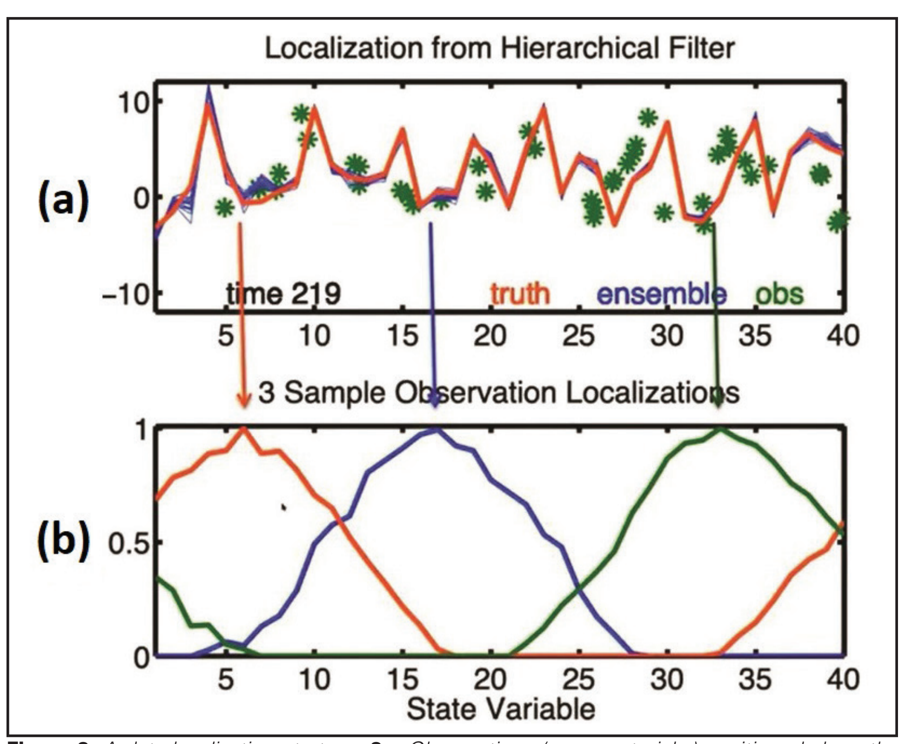


Figure 2. *A data localization strategy.* **2a.** *Observations (green asterisks) positioned along the horizontal axis.* **2b.** *Localization weighting for three of these observations. Figure courtesy of Jeff Anderson.*

Computational weather models are inherently inaccurate because of finite-precision arithmetic, if nothing else. Anderson and his colleagues mimicked model error by changing the forcing constant F in the Lorentz 96 system. Subsequent simulations demonstrated the success of an adaptive approach for managing model bias.

Other work has explored non-Gaussian priors, which might arise from observations of a trace chemical's concentration and can never be negative. Methods to reduce some of the additional restrictions and limitations of ensemble filtering—such as the need for regression to increment state variables—are currently under study.

Despite the progress that he reported, Anderson made clear that enabling ensemble filtering on a geophysical scale is far from done and dusted. Future directions might include particle filters, their formidable computational costs notwithstanding. "Lots of fun is still left in merging ensemble and particle filters," Anderson said.

Paul Davis is professor emeritus of mathematical sciences at Worcester Polytechnic Institute.

Obituary: Andreas Griewank

By Andrea Walther and Uwe Naumann

Andreas Griewank passed away suddenly and unexpectedly on September 16, 2021, at the age of 71. We mourn the loss of an outstanding, internationally recognized mathematician whose groundbreaking contributions in algorithmic/automatic differentiation (AD) shaped the field of modern-day optimization.

Andreas was born in Kassel, Germany, on January 26, 1950. After finishing high school at the Albert-Schweitzer-Gymnasium in 1968, he enrolled at the Technische Universität in Clausthal-Zellerfeld and began studying mathematics and physics. Andreas moved to Freiburg in 1972 and pursued mathematics, physics, and economics at Albert-Ludwigs-Universität. He graduated with distinction in 1975 with a diploma thesis on affine linear automata under the direction of Lutz Eichner. Andreas then joined the Computer Centre and Department of Computer Science at the Australian National University in Canberra, where he completed his master's degree in 1977 with a thesis titled *A Generalized Descent Method for Global Optimization*. He remained in Canberra for his doctorate—advised by Richard P. Brent and Michael R. Osborne—and received his Ph.D. in 1980 for a dissertation on *Analysis and Modification of Newton's Method at Singularities*.

After earning his Ph.D., Andreas worked with Michael J.D. Powell as a postdoctoral researcher at the University of Cambridge. In 1982, he became an assistant professor at Southern Methodist University in Texas; he was promoted to tenured associate professor in 1986. From 1987 to 1993, Andreas worked as a senior mathematician at Argonne National Laboratory. He agreed to a professorship at Technische Universität (TU) Dresden in 1993 that included directorship of the Institute for Scientific Computing. He then spent a sabbatical at INRIA Sophia Antipolis in France from 1998 to 1999. After returning to TU Dresden, Andreas accepted a Matheon professorship at the Humboldt-Universität zu Berlin in 2003. He later served as director of the Institute of Mathematics at Humboldt. Following retirement in 2015, Andreas devoted his energy to the newly founded Universidad Yachay Tech in Ecuador, where he was dean of the School of Mathematical and Computational Sciences from 2015 to 2019.

Andreas Griewank, 1950-2021. Photo courtesy of Philipp Griewank.

Andreas is widely considered to be the godfather of AD. He began working in this research area in the early 1980s and quickly became the driving force for AD's development and the field's leading proponent. His broad perspective covered many areas—including theoretical foundations, efficient algorithms, and the development of mature software tools—and the resulting applications touched a myriad of domains in science, engineering, and economics. In fact, his book *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation*[1] (published by SIAM with coauthor Andrea Walther) is already in its second edition and is still a standard reference on the subject.

[1] https://my.siam.org/Store/Product/viewproduct/?ProductId=1005

Andreas continuously made important contributions to the design and analysis of nonlinear optimization algorithms. Here we mention just a few of his accomplishments in the context of very different aspects of nonlinear optimization. First, Andreas and Philippe Toint jointly developed the idea of partial separability. This structural property is ubiquitous in optimization problems and can be exploited to greatly improve algorithm efficiency. Andreas also advanced the convergence theory of Newton and quasi-Newton methods in multiple settings, including the infinite dimensional setting and the degenerate setting — in which the Hessian is singularly the optimum. These topics were the subject of a frequently cited series of papers that form the basis of ongoing research.

The so-called "Griewank function," which serves as an academic test function in the field of global optimization, is another one of Andreas' contributions. This function sees widespread use within the global optimization community and is the subject of renewed interest, as non-convex optimization minimizes objectives in data analysis applications like deep learning.

Andreas made significant service contributions to several different communities, and his scientific work was routinely marked by an abundance of ideas and infectious enthusiasm. He organized numerous conferences and workshops around the world, several of which were under the aegis of SIAM. These meetings often aimed to connect academic researchers with practitioners in the areas of AD and nonlinear optimization, a goal that is consistent with SIAM's outreach to industry and practice.

In 2001, Andreas received the Max Planck Research Award. He was named a Fellow of SIAM in 2017. Andreas also served SIAM in several different leadership roles; he was elected to the SIAM Council in 2013 and was a member of the Board of Trustees from 2015 to 2016.

Andreas was always passionate about promoting young researchers; he supervised 23 doctoral students and numerous master's students during his time in academia. He also had a special interest in supporting mathematical education in developing countries. In addition to his commitment at Yachay Tech in Ecuador, Andreas was actively engaged in groups such as the European Mathematical Society's Committee for Developing Countries and the International Mathematical Union's Commission for Developing Countries.

His quick death without any significant prior health issues fits Andreas as we knew him. He was restless until the end and always full of energy for life and new mathematical developments. He lived a very fulfilling life and will be severely missed.

Andrea Walther is the MATH+ Professor for Mathematical Optimization at the Humboldt-Universität zu Berlin. Her research interests are in the fields of adjoint-based optimization, nonsmooth optimization, and algorithmic differentiation. Uwe Naumann is a professor of computer science at RWTH Aachen University. His research is inspired by algorithmic differentiation with a focus on combinatorial problems, software engineering, and adjoint numerical methods.

Math Modeling Workshops for High School Students

By Wesley Hamilton and Keshav Patel

The University of Utah's Department of Mathematics recently launched a new program[1] that introduced high school students in Salt Lake County and the surrounding areas to mathematical modeling. The program, which ran throughout September and October 2021, sought to prepare students for mathematical modeling competitions like the High School Mathematical Contest in Modeling (HiMCM)[2] and MathWorks Math Modeling Challenge (M3 Challenge),[3] a program of SIAM.

Over the course of eight weeks, participating students learned the fundamental concepts of mathematical modeling frameworks—including difference/differential equations, regression, and parameter fitting—as well as basic programming and writing skills. Regardless of whether they ultimately formed teams to compete in the upcoming modeling competitions, all students developed useful technical and problem-solving expertise that will help them as they continue to hone their mathematical abilities.

Due to the workshop's online format, high school students from all over Utah were able to partake. The Utah State Board of Education even advertised the workshop in its twice-monthly science, technology, engineering, and mathematics (STEM) educator newsletter. As such, the program had attracted motivated students from all types of mathematical backgrounds by the time it commenced in early September.

Participants met virtually for two hours each week, ultimately leading up to HiMCM in November. The first hour of each session consisted of an active lecture wherein volunteers familiarized students with programming techniques, data analysis, model fitting, sensitivity analysis, and report writing, among other topics. These lectures engaged attendees with discussion-based activities and group coding sessions in Python via Google Colab[4] (see Figure 1). Presenters had creative control over their sessions and often incorporated their own modeling experiences into the conversation. For example, Julie Sherman (University of Utah) adapted a discussion about sea turtle population dynamics from a previous workshop at the University of Minnesota[5] into a larger conversation about difference/differential equations (see Figure 2, on page 6).

During the second hour of each session, students split into small teams and worked on real-world problems from previous M3 Challenge years. These questions addressed issues like food waste and insecurity[6] and the rise of e-cigarette use among teenagers.[7] Students were encouraged to make and justify assumptions, define relevant variables to serve as model inputs and outputs, and discuss the types of mathematical frameworks that would effectively encompass the scenarios. In just an hour, participants

See **Modeling Workshops** on page 6

[1] https://www2.math.utah.edu/high-school-program/math-model-workshop.php
[2] https://www.comap.com/highschool/contests/himcm/index.html
[3] https://m3challenge.siam.org
[4] https://colab.research.google.com
[5] https://mathinsight.org/assess/solveit/sea_turtle_introductory_example
[6] https://m3challenge.siam.org/archives/2018/problem
[7] https://m3challenge.siam.org/archives/2019/problem

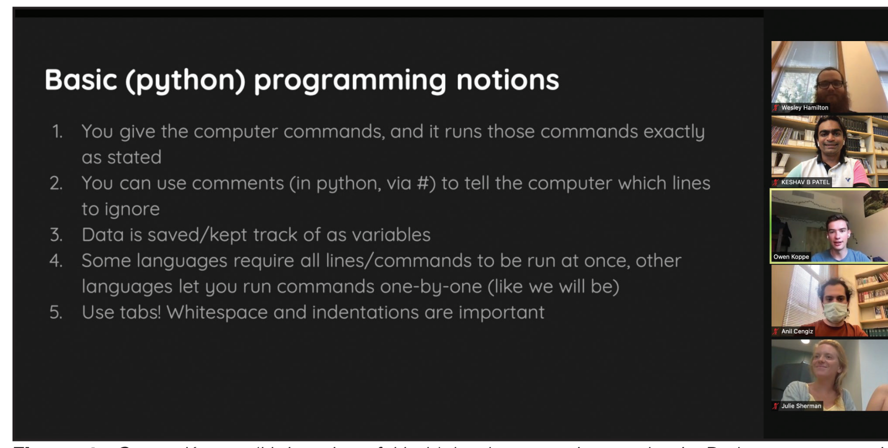


Figure 1. *Owen Koppe (University of Utah) leads a session on basic Python programming during the University of Utah's math modeling workshop for high school students, which took place in the fall of 2021.*

Medicine

Continued from page 1

of life on Earth, so experiments that involve Wnt in one organism—such as a fruit fly or *Xenopus* frog—can act as experimental proxies for Wnt in humans. "The Wnt pathway experiences dysfunction in 90 percent of colorectal cancer," Harrington said, noting that colorectal cancer is among the most common cancers in men and women. "So it's really important to understand the underlying molecular mechanisms."

Researchers have proposed multiple models for the processes that shuttle proteins around cells and the degradation of the protein β-catenin, which mediates the Wnt signaling pathway. Each model involves multiple coupled nonlinear differential equations, where the equations govern the amounts of particular protein species in the pathway in terms of their chemical reaction, formation, and destruction. Harrington and her collaborators incorporated features from these existing models to build a "shuttle model" that characterizes the entire pathway and has 19 species of proteins and 31 parameters [2].

In general terms, these models consist of 19 polynomial differential equations, one for each species $x_i \in \{x\}$:

$$\dot{x} = YA_k\Psi(x).$$

Here, Y is a matrix of integers that reveal the number of occurrences for each reaction (the stoichiometric parameters, in chemical terms); A_k is a matrix containing the parameters that govern reaction, growth, and decay rates; and $\Psi(x)$ includes the dynamics of the species concentrations. Writing out all 19 equations for the shuttle model would take up an unreasonable amount of space; fortunately, these equations are simplified by the fact that not all species interact with each other. The following two equations demonstrate how they look in practice:

$$\dot{x}_1 = k_1x_1 + k_2x_2,$$
$$\dot{x}_2 = k_1x_1 - (k_2 + k_{26})x_2 + k_{27}x_3 - k_3x_2x_4 + (d_4 + k_5)x_{14}.$$

Rather than use the matrix indices, the nonzero parameters are labeled k_α.

As with other dynamical systems, the fixed points of the shuttle model (where $\dot{x}_1 = 0$) characterize important behaviors like cell death. However, the number of equations and parameters make extraction of these behaviors extremely complicated, apart from special parameter choices.

To circumvent this issue, Harrington and her colleagues employed chemical reaction network theory (CRNT): a mathematical framework in applied algebraic geometry that identifies equations' qualitative features without having to establish parameter values. At the fixed points, the differential equations become polynomial equations for which only real-valued positive protein amounts are biologically realistic. Harrington combined CRNT, numerical algebraic geometry, and matroid theory to identify stable states for the Wnt pathway in the shuttle model and design future experiments for model testing.

Despite this method's complexity, Harrington does not pretend that it corresponds perfectly to real-world processes in cells. "A biologist abstracts a signaling network from the messy molecular interactions inside a cell," she said. "But it's not a true network in a mathematical sense. One of the assumptions we make when looking at these systems using chemical reaction networks is that the rate at which a reaction proceeds is proportional to the product of the species' concentration."

Though Harrington's earlier work attempted to analyze a single pathway at steady state, her team is becoming increasingly interested in joining multiple pathways and studying them with transient time-course data [1]. New mathematics and methods are necessary if scientists hope to better represent and understand the true biology.

Topologically Speaking

The aforementioned dynamical models that describe Wnt only involve changes in time. However, many important biological applications require consideration of variations in *space* as well. For example, researchers can map angiogenesis (the growth of blood vessels) in tumors, which offers insight as to how cancers receive nutrients and how the immune system fights back.

Angiogenesis in cancer has intrigued Harrington ever since she conducted a summer research project as an undergraduate student at the University of Massachusetts. "We were running simulations on the big computing cluster at UMass, tracking the angiogenesis and growth of new vasculature towards a tumor," she said. Harrington then described how her group judged the results of their simulations by visual inspection rather than with a more mathematically rigorous method. "I thought that there must be a better way to quantify this," she added.

Yet at the time, the simulations taxed the available computers, thereby limiting the kinds of simulations that Harrington desired. Thankfully, improvements in algorithms and computers over the following decades have moved these seemingly impossible problems into the realm of solvability.

Real-world angiogenesis data tends to be very noisy, which complicates the analysis of spatial models and data. Harrington's team addressed this issue by adapting topological data analysis (TDA)—specifically the persistent homology algorithm—to create a TDA pipeline that analyzes angiogenesis (see Figure 1, on page 1). More recently, she showcased a statistical topology technique to determine diferent spatial patterns of immune cells that infiltrate a tumor.

Persistent homology allows researchers to simultaneously examine the topology of a set of data points on multiple scales so that the technique can handle the fuzziness of experimental data. In fact, fuzzy data that also contained outliers helped motivate the development of multi-parameter persistent homology. "Persistent homology landscapes are a statistical way to compare when you have more than one parameter," Harrington said. She used the example of the density of immune cells relative to regions where the tumor is deprived of oxygen (hypoxia). She and her collaborators were even able to distinguish between different types of immune cells with these methods [4].

Both the algebraic and topological approaches have their advantages, but Harrington finds that their power comes through in flexibility. "I'm not using one approach and trying to make everything fit into that," she said. "As we go from a couple of molecular species to groups of cells or tissues, the problems become spatiotemporal and the mathematical approaches need to change — especially when we want to include real data with noise."

References

[1] Gross, E., Harrington, H., Meshkat, N., & Shiu, A. (2020). Joining and decomposing reaction networks. *J. Math. Biol.*, *80*, 1683-1731.

[2] MacLean, A.L., Rosen, Z., Byrne, H.M., & Harrington, H.A. (2015). Parameter-free methods distinguish Wnt pathway models and guide design of experiments. *PNAS*, *112*(9), 2652-2657.

[3] Stolz, B.J., Kaeppler, J., Markelc, B., Mech, F., Lipsmeier, F., Muschel, R.J., …, Harrington, H. (2020). Multiscale topology characterises dynamic tumour vascular networks. Preprint, *arXiv:2008.08667*.

[4] Vipond, O., Bull, J.A., Macklin, P.S., Tillmann, U., Pugh, C.W., Byrne, H.M., & Harrington, H.A. (2021). Multiparameter persistent homology landscapes identify immune cell spatial patterns in tumors. *PNAS*, *118*(41), e2102166118.

Matthew R. Francis is a physicist, science writer, public speaker, educator, and frequent wearer of jaunty hats. His website is BowlerHatScience.org.

Justice, Equity, Diversity, and Inclusion in Applied Mathematics for All

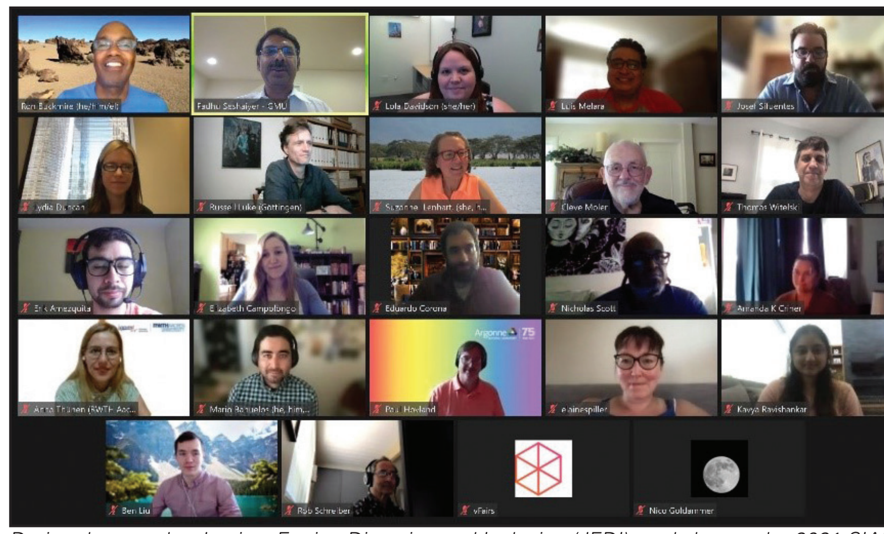
Interactive Session at AN21 Addresses Racial Equity

By Padmanabhan Seshaiyer and Ron Buckmire

To enhance the mathematical community's understanding of racial equity in applied mathematics research and education, SIAM held its first Justice, Equity, Diversity, and Inclusion (JEDI) workshop during the 2021 SIAM Annual Meeting (AN21),[1] which took place virtually in July. Ron Buckmire—SIAM's Vice President for Equity, Diversity, and Inclusion (EDI)—and Padmanabhan Seshaiyer—chair of SIAM's Diversity Advisory Committee (DAC)—hosted and moderated the two-part series. DAC members Suzanne Lenhart (University of Tennessee), Ami Radunskaya (Pomona College), Josef Sifuentes (University of Texas Rio Grande Valley), and Luis Melara (Shippensburg University) helped facilitate the session.

The interactive nature of this inaugural event allowed participating students, faculty, and staff to discuss equitable opportunities to ultimately transform institutional practices at multiple levels and settings. The JEDI workshop integrated best practices in mathematical research, outreach, and engagement programs with retention and recruitment strategies that pertain to the most pressing questions and compelling opportunities for advancing racial equity in applied mathematics education. We strove to reach beyond the current paradigms that contribute to persistent racial inequalities in the field of applied math.

The session actively engaged the audience with brainstorming projects via Google Jamboard and Google Slides. Attendees partook in a four-corner pedagogical activity called "Notice and Wonder," wherein they examined four images that illustrated the importance of the differences between the words "inequality," "equality," "equity," and "justice" (see Figure 1). For example, the notion of *equality* addresses some systemic concerns by providing the same applied mathematics resources and opportunities to all students, but *equity* goes further by recognizing that each individual has different needs. This realization highlights the value of customized resources and opportunities. Participants also realized that while the concept of equity may address imbalanced social systems in the applied math community, *justice* would mean that these systems are fixed in a way that leads to long-term, sustainable, and unbiased access to resources and opportunities.



During the two-day Justice, Equity, Diversity, and Inclusion (JEDI) workshop at the 2021 SIAM Annual Meeting, which took place virtually in July, participants engaged in interactive exercises to advance their understanding of racial equity in applied mathematics research and education.

Next, session attendees identified similarities and differences between justice and EDI in applied mathematics with a Venn diagram exercise. Their responses clearly highlighted the need for additional conversation within the community. Relevant topics for further discussion include mentorship, systemic racism and culturally responsive teaching practices, increased representation, equitable assessment and evaluation, and economic and racial disparities.

During the final part of the workshop, participants presented ideas or actions that they—and/or their units/departments/universities—can take (or are already taking) to promote JEDI practices. The response was overwhelming. Suggestions included collecting evidence of disproportionate representation in the number of awarded bachelor's degrees, investigating underrepresentation of women and racial and ethnic minorities, providing implicit bias training, amending hiring practices, and re-envisioning speaker selection at conferences. Discussions also highlighted the impact of unconscious gender bias, which is influenced by cultural stereotypes, affects how individuals evaluate and treat one another, and continues to impact both university and workplace environments. Ultimately, this dialogue built effective practices to combat systemic barriers to opportunities and address access, retention, and success in applied mathematics education.

Another pedagogical practice called "KWL: **K**now, **W**ant, **L**earn" prompted attendees to voice "What they *know*," "What they *want* to know," and "What they *learned*." This exercise allowed individuals to identify and share current challenges and opportunities that relate to JEDI in the applied math community.

According to the exit survey, more than 92 percent of attendees felt that the session had a positive impact. One participant mentioned that they learned "how to explain/talk about race versus ethnicity." Others reflected on connections that they

See **Justice** *on page 7*

[1] https://www.siam.org/conferences/cm/conference/an21

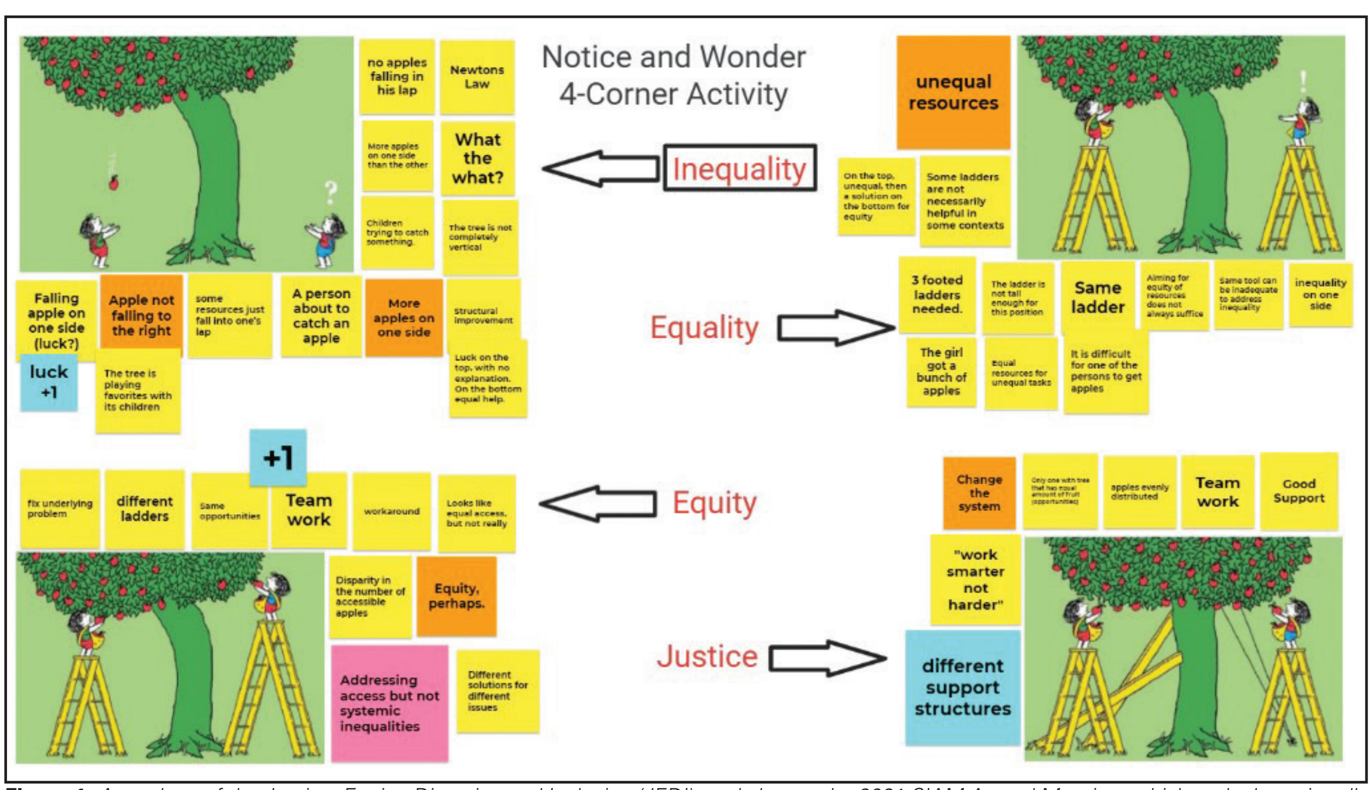


Figure 1. *Attendees of the Justice, Equity, Diversity, and Inclusion (JEDI) workshop at the 2021 SIAM Annual Meeting, which took place virtually in July, partook in a collaborative "Notice and Wonder" pedagogical activity that demonstrated the need for systems that provide everyone with access to tools and opportunities. Figure courtesy of Padmanabhan Seshaiyer.*

Exploring the Radon Transform and the Field of Medical Imaging

By Peter Kuchment

The following is a brief reflection from the author of The Radon Transform and Medical Imaging,[1] *which was published by SIAM in 2013 as part of the CBMS-NSF Regional Conference Series in Applied Mathematics. The text surveys the main mathematical concepts and techniques that drive both well-established imaging modalities and developing methodologies. It explores a variety of concepts that pertain to medical imaging, including stability, inversion, incomplete data effects, and the role of interior information.*

The Radon Transform and Medical Imaging addresses the topics of 10 lectures that I delivered during the 2012 NSF-CBMS Conference on Mathematical Methods of Computed Tomography.[2] Tomography aims to produce a picture of the internal slices of a nontransparent body, such as a patient's physique, an airplane wing, or the Earth itself. It does so by sending signals—ultrasound, X-rays, electrical currents, light, and so forth—to penetrate the body in question, then recording the outcome. In other words, the body acts like a "black box" into which practitioners send signals and examine the resulting outputs. We must then recover the hidden mechanism from these observations.

However, we cannot address the problem with "black box" generality. Signal propagation (and thus observable output) is governed by a "hidden" partial differential equation (PDE). Although we usually possess some general knowledge about the equation, we are often not sure of its coefficients. The coefficients therefore comprise the overall picture that we seek. This setup is called an *inverse problem*; we know the answers but not the questions that are hidden inside the box. One famous example of such a problem is evident in Douglas Adams' *The Hitchhiker's Guide to the Galaxy*, which discusses the "ultimate question of life, the universe, and everything." The answer, calculated by a supercomputer in a matter of 7.5 million years, was 42. But no one knew what the question was!

FROM THE SIAM BOOKSHELF

The wide range of important applications for computed tomography (CT) explains why it has been attracting the attention of medical practitioners, mathematicians, physicists, engineers, geophysicists, and other scientists for the last half-century. In addition to its usefulness, CT boasts a whole world of mathematical techniques and problems that delight both pure and applied mathematicians: commutative and non-commutative Fourier (harmonic) analysis; differential equations; geometry (integral, differential, and algebraic); complex analysis (in several variables); microlocal analysis; group representation theory; discrete mathematics; probability theory and statistics; and numerical analysis.

The audience of the 2012 NSF-CBMS conference encompassed a wide range of attendees, from graduate students to established researchers. I therefore attempted to keep the exposition as self-contained as possible; in most cases, experts should skip the background information in the appendices to avoid feeling patronized. I also tried to minimize technicalities and instead emphasize the main mathematical ideas and intuitions. Readers can develop the details to their full extent or find them in the literature (the text contains pointers to help them do so) — this framework explains the book's rather colloquial style. Here I briefly describe the contents of *The Radon Transform and Medical Imaging*.

The first part of the text introduces readers to medical imaging and a variety of relevant problems, including image reconstruction (the goal of the book), processing, and interpretation. It then explains the notion of CT and provides a classification into transmission, emission, and reflection types. I also present various application areas beyond medicine, such as industry; homeland security; geology, geophysics, and seismology; radar, sonar, and lidar; and even archeology.

The text then lists the milestones in the subject's more than 100-year history and discusses contrast, stability, and resolution. Since nearly all relevant methods involve the recovery of a known PDE's unknown coefficients, the early chapters address the types of equations that arise in several common CT techniques. Knowledge of the type of equation (hyperbolic, parabolic, elliptic, etc.) offers users important information about the kind of resolution they can expect.

The book's second part overviews the basic mathematics of the most familiar traditional techniques: X-ray CT and emission tomography. Chapters five and six tackle topics like inversion, reconstruction stability, and artifacts and their microlocal origins. This segment proceeds to present the main notions of CT (such as backprojection) and derives important formulas (e.g., inversion formulas, the Fourier slice theorem, and range conditions).

Chapter seven examines the important issue of artifacts in reconstruction and the microlocal approaches to understand and predict them. While learning microlocal analysis in its full capacity is a daunting task for a beginner, microlocal analysis results and tests are very easy to use. Chapter nine then briefly surveys numerical techniques, which could fill a book by themselves, and chapter 10 briefly describes other popular medical imaging modalities like magnetic resonance imaging, optical tomography, and ultrasound.

The third part of the text introduces an abundance of newly developed, so-called hybrid techniques. Since each known modality suffers from some drawbacks (e.g., low resolution or low contrast), the hybrid methods strive to combine them to alleviate their individual faults and capitalize on their advantages. The main concentration here is thermo-(opto-) acoustic tomography, ultrasound modulation of electrical and optical tomography, and problems with "internal information."

Finally, the book's fourth and final part consists of appendices that address the harmonic analysis, differential equations, and functional analysis tools that I utilize throughout the text.

The Radon Transform and Medical Imaging does not assume that readers are familiar with CT, which makes it accessible to a wide audience of specialists and nonexperts alike. Graduate students and researchers in mathematics, engineering, and physics who wish to learn more about medical imaging will all benefit from reading this text.

Enjoy this passage? Visit the SIAM Bookstore to learn more about The Radon Transform and Medical Imaging[3] *and browse other SIAM titles.*

Peter Kuchment is a University Distinguished Professor of Mathematics at Texas A&M University. His research areas of interest include spectral theory, partial differential equations, mathematical physics, inverse problems, and imaging. Kuchment is also a SIAM Fellow.

[1] https://my.siam.org/Store/Product/viewproduct/?ProductId=25096489

[2] https://fermat.uta.edu/cbms2012/index.html

[3] https://my.siam.org/Store/Product/viewproduct/?ProductId=25096489

Modeling Workshops

Continued from page 4

developed an initial plan of action to tackle the competition questions. Attendees appreciated the applicability of these tools to a wide variety of real-world situations. "Prior to the workshop, I never thought of attacking economic problems like world hunger using math," Caelum van Ipselen, a junior at Rowland Hall in Salt Lake City, said. "It was intriguing to find out how mathematics could offer solutions to these problems."

In the second hour of the following week's workshop, volunteers walked students through winning solutions as well as a stratified set of anonymous non-winning solutions. Students then analyzed these solutions and discussed the validity of assumptions, the choice of variables, and the structure of the mathematical models. By the end of the workshop series, participants could think critically about their own work, make stronger arguments for their assumptions, create more robust models, and recognize the important organizational aspects of a competitive contest submission.

The workshop's content was based on two modeling handbooks that are published by SIAM: *Math Modeling: Getting Started and Getting Solutions*[8] and *Math Modeling: Computing and Communicating*.[9] These resources are freely available online. Though this program was meant to prepare students for HiMCM (sponsored by the Consortium for Mathematics and Its Applications), much of the material came from SIAM's M3 Challenge — an annual mathematical modeling competition for high school juniors and seniors in the U.S. and sixth form students in England and Wales. Teams of up to five students have 14 hours to generate mathematical models for a given set of questions and communicate their results in a comprehensive written paper. The Challenge problem generally contains a three-part, data-focused question; students may choose to utilize statistical analysis and/or programming knowledge of MATLAB, Python, or any other software (or combination of software) to predict future events. The competition awards annual scholarship prizes of more than $100,000 and designates additional extra credit scholarship awards for teams who choose to submit code with their solution papers. M3 Challenge aims to motivate young people to study and pursue careers in applied mathematics, computational science, and technical computing by bringing together students and applied math practitioners.

Historically, not many teams from the Salt Lake County area have participated in M3 Challenge or HiMCM, so the workshop increased awareness of mathematical modeling for local students. For instance, it provided van Ipselen's first glimpse into the world of math modeling and its associated competitions. "It was great to receive pragmatic advice from former competitors," he said. "They invariably gave experience-based answers to my questions, and I walked out confident in what I needed to do well." Van Ipselen competed in HiMCM in November 2021 and plans to partake in future competitions. Some educators who are interested in incorporating mathematical modeling into their classrooms have also expressed interest in partnering with the program leads on future projects.

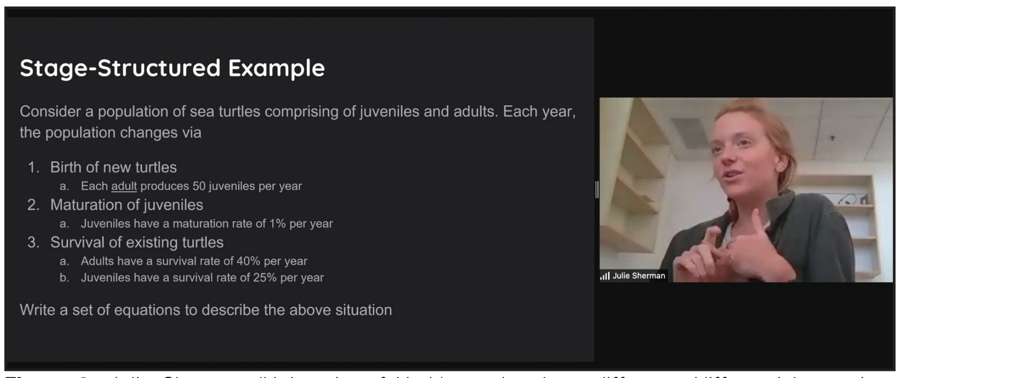


Figure 2. *Julie Sherman (University of Utah) speaks about difference/differential equations with an application to sea turtle population dynamics during the University of Utah's math modeling workshop for high school students, which took place in the fall of 2021.*

After the program concluded, planning immediately commenced for the next workshop series, which will take place in mid-January to February — just before M3 Challenge. With the help of SIAM and MathWorks, this forthcoming workshop will adapt hands-on coding examples into MATLAB (MathWorks' signature software) and reach a larger group of motivated high school students and their teachers/coaches. Sessions from the program will be recorded and all code and other materials will be available online so that interested students can learn about—and start coding for—math modeling. High school and sixth form students, as well as secondary teachers who might serve as coaches, can register for the January-February 2022 workshop series and access information about session dates and times online.[10]

Acknowledgments: Wesley Hamilton (University of Utah) and Sheina Rodriguez (Orlando Math Circle) prepared the program's original slides with the help of Kate Daftari, Austin Ferguson, Andrew Ford, Scott Hallyburton, Anne Talkington, and Wenzhong Wang (all of the University of North Carolina at Chapel Hill). Lindsey Henderson (Utah State Board of Education) graciously advertised the workshop to her network of Utah STEM educators.

Moreover, this workshop could not have occurred without the time and effort of the following faculty and applied mathematics students at the University of Utah: Anil Cengiz, Owen Koppe, Xiao Shen, Julie Sherman, and Nathan Willis.

Wesley Hamilton and Keshav Patel created and organized the University of Utah's math modeling workshop for high school students. Wesley Hamilton is a Wylie Assistant Professor at the University of Utah. He also serves on the SIAM Education Committee. Keshav Patel is an NSF Graduate Research Fellow at the University of Utah.

[8] https://m3challenge.siam.org/sites/default/files/uploads/siam-guidebook-final-press.pdf

[9] https://m3challenge.siam.org/sites/default/files/uploads/siam-technical-guidebook-web.pdf

[10] https://siam.zoom.us/webinar/register/WN_wPNBAoVlQMKvIF4ENkkkzA

Recapping the 44th SIAM Southeastern Atlantic Section (SEAS) Conference

By Yanzhao Cao and Junshan Lin

The 44th SIAM Southeastern Atlantic Section (SEAS)[1] Conference took place in September 2021 at Auburn University with a hybrid modality. The two-day conference,[2] which was initially scheduled for March 2020, attracted more than 200 participants from the U.S. and five other countries. Roughly 70 people attended the meeting in person, and the remainder joined virtually on Zoom.

The conference program included four plenary talks, 43 parallel minisymposium sessions, four groups of contributed presentations, and a poster session/reception. Oscar Bruno (California Institute of Technology), Jianfeng Lu (Duke University), James Nagy (Emory University), and Juan M. Restrepo (Oak Ridge National Laboratory) served as the plenary speakers. The minisymposia addressed a wide variety of topics that ranged from emerging developments in machine learning and data science to recent advances in theories and numerical methods for partial differential equations and their applications.

All of the talks were livestreamed over Zoom with a physical audience in the conference rooms on campus.

Many in-person participants expressed their appreciation to SIAM for the opportunity to interact with academic colleagues and collaborators face to face after nearly two years of virtual conferences. The hybrid nature of the SIAM-SEAS Conference was made possible by the cutting-edge, high-tech classroom facilities at Auburn University. The conference organizers would also like to thank SIAM as well as the Department of Mathematics and Statistics and the College of Sciences and Mathematics at Auburn University for their ardent support in bringing this long-delayed event to fruition. They would be happy to share their experiences surrounding the hybrid conference modality with anyone in the SIAM community who is interested in organizing future meetings with similar setups.

Yanzhao Cao is the Don Logan Endowed Chair of Mathematics at Auburn University. He currently serves as president of the SIAM Southeastern Atlantic Section. Junshan Lin is an associate professor of mathematics at Auburn University. He currently serves as vice president of the SIAM Southeastern Atlantic Section.

[1] https://www.siam.org/membership/sections/detail/siam-southeastern-atlantic-section-siam-seas
[2] http://www.auburn.edu/cosam/siam/index.htm

Alessandro Veneziani of Emory University delivers a presentation on model reduction in cardiovascular mathematics during the 44th SIAM Southeastern Atlantic Section (SEAS) Conference, which took place in September 2021. Photo courtesy of Yanzhao Cao.

Justice

Continued from page 5

made in the workshop and recognized the danger of making assumptions about ancestry because everybody's personal experiences and history are different.

In addition to inspiring stimulating discussion, the JEDI workshop at AN21 provided training for applied mathematics educators and graduate students who wish to promote awareness of implicit biases, which can take many forms. For instance, implicit bias may involve assumptions about students' learning behaviors and capabilities for academic success based on their identities and/or backgrounds. The session also created a platform for healthy discussion wherein educators could self-assess their own biases, learn to develop formal and informal reflective teaching strategies, and work to foster an inclusive applied mathematics classroom climate to enhance their teaching and mentoring practices.

Gerardo Hernández-Dueñas, president of the Mexico Section of SIAM, was inspired by the JEDI session and hosted a similar workshop at the Annual Meeting of the Mexican Geophysical Union[2] in November 2021 on diversity, ethics, equity, and inclusiveness in the context of best practices in academic and geoscience organizations. "At MexSIAM, we believe that these are fundamental topics to our community," Hernández-Dueñas said. "I found the tools you presented very impressive and efficient to engage the mathematics community to learn and share new ideas about JEDI." Another JEDI collaborative session[3] with additional speakers will take place at the upcoming 2022 Joint Mathematics Meetings in Seattle, Wash.[4]

Padmanabhan Seshaiyer is a professor of mathematical sciences at George Mason University and chair of the SIAM Diversity Advisory Committee. He works in the broad area of computational mathematics, data science, biomechanics, design and systems thinking, and STEM education. Seshaiyer also serves as vice chair of the U.S. National Academies Commission on Mathematics Instruction. Ron Buckmire is a professor of mathematics and Associate Dean for Curricular Affairs at Occidental College. He publishes research in numerical analysis, mathematics education, and data science. Buckmire is also the Vice President for Equity, Diversity, and Inclusion at SIAM.

[2] https://www.raugm.org.mx

[3] https://www.jointmathematicsmeetings.org/meetings/national/jmm2022/2268_program_mini7.html
[4] https://www.jointmathematicsmeetings.org

Making Mathematics Meaningful on the 2021 SciFest All Access STEM Stage

By Padmanabhan Seshaiyer

The USA Science & Engineering Festival (USASEF)[1] continued its bi-annual exposition with the *SciFest All Access* event[2] from October 18-24, 2021. SciFest All Access offered a free virtual science, technology, engineering, and mathematics (STEM) exposition for K-12 students, educators, and their families. SIAM has participated in USASEF since it commenced more than a decade ago. This year's events included STEM learning experiences, online field trips, exhibit zones, scavenger hunts, student project displays, and "STEM Stage" performances by well-known speakers on a variety of topics.

Once again, USASEF invited me to speak on the virtual STEM Stage about "Making Mathematics Meaningful." I was glad to engage listeners with important pedagogical frameworks and practices—such as computational thinking and mathematical modeling—that serve as valuable problem-solving strategies for the application of mathematics to real-world problems. Participants learned how to navigate complex scenarios by employing strategies like problem decomposition, pattern recognition, abstraction, and algorithmic thinking. Specifically, I described the use of computational and modeling skills to both solve textbook problems (such as calculating the number of squares on a chessboard) and address global challenges (such as food waste in school cafeterias). Figure 1 illustrates these concepts.

Figure 1. *At the SciFest All Access event, which took place virtually in October as part of the USA Science & Engineering Festival, Padmanabhan Seshaiyer spoke about the value of computational and modeling techniques when calculating the number of squares on a chessboard or estimating the amount of food waste in school cafeterias. Figure courtesy of the author.*

To tackle these types of problems, I introduced simple tools like Fermi problem-solving approaches; such methods encourage users to think algorithmically and estimate answers via back-of-the-envelope calculations [3]. For example, estimates of food that is wasted by K-12 students in pounds per week can help one calculate larger approximations of waste based on the number of students per class, number of classes per school, number of schools per district, number of districts per state, and number of states per country. Attaching a dollar value to a wasted food item then allows one to apply simple calculations to roughly determine the amount of food that is wasted in the country each week in school cafeterias alone. These tactics build students' understanding and appreciation of mathematics when they wonder, *"When am I ever going to use this?"* I also utilized games to engage students in mathematical modeling and prompted them to think about the meaning of "best" in terms of optimal problem solving.

In addition, I had the exciting opportunity to speak to K-12 teachers and university educators about meaningful mathematics through USASEF's "Spark of STEM Coffee Break Series." This free online series provides 15-minute "sparks" of inspiration for STEM teaching in the classroom and is meant to inspire and equip teachers with innovative STEM learning tools. Future vignettes will take place on Thursdays in December 2021 and are available on demand. Educators who are interested in participating can register online.[3]

Ultimately, the goal of such outreach and engagement activities is to help students ask bold questions and persist through creatively imagined and calculated solutions. These exercises also provide students with the tools to collect, interpret, visualize, and predict data, which is an important transferable competency. Most institutions are beginning to recognize the importance of engaging students beyond a *content-based* education and are thus seeking to provide a *competency-based* education that includes data, computation, and global

See **SciFest All Access** *on page 10*

[1] https://usasciencefestival.org
[2] https://scifest.vfairs.com
[3] https://usasciencefestival.org/sos-coffee-break-series-2021

ATTENTION SIAM Math Modelers!

GOT A PROBLEM?

SIAM is Seeking Problem Ideas for Math Modeling Competition

MathWorks Math Modeling Challenge
A program of SIAM

$100,000 (£75,000) in SCHOLARSHIPS

What is M3 Challenge?

MathWorks Math Modeling (M3) Challenge is an Internet-based, applied mathematics contest that takes place each year in February or March. High school juniors and seniors in the U.S. and sixth form students (age 16-19) in England and Wales may form and enter up to two teams of three to five students each per school. Teams are given 14 hours to solve an open-ended, applied math-modeling problem related to a real-world issue. Working collaboratively, students use math modeling to represent, analyze, make predictions and provide insight into current world issues.

Past topics addressed issues such as substance abuse, food insecurity, climate change, car sharing, and modeling the cost, needs assessment, and placement of towers for maximizing access to the internet.

The goal of the Challenge is to motivate students to study and pursue careers in STEM disciplines, especially applied mathematics, computational science, data science, and technical computing. The problem is revealed to students only after they login on their selected Challenge day. Solutions are judged on the approach and methods used and the creativity displayed in problem solving and mathematical modeling. Extra credit in the form of technical computing scholarship awards is available for teams who opt to submit code.

Winners receive scholarship prizes totaling $100,000 (£75,000). Registration and participation are free.

Problem structure

Within the problem statement, there should be three questions for teams to answer:

- Question One: A warm up — every serious team can answer.
- Question Two: The guts — framed so that every team can have some success and many teams will cover it well.
- Question Three: The discriminator — many teams will do something, while only a few will have striking results.
- Data — data that is provided or easily found is desirable to encourage students to use coding and technical computing in solution papers.

Honoraria

- $50 for problems found suitable to add to the M3 Challenge problem reserve "bank"
- $500 for problems selected from the reserve bank to be used as "the" Challenge problem

Required problem characteristics

- Accessibility to high school/sixth form students
- Suitability for solution in 14 hours
- Possibility for significant mathematical modeling
- Topic of current interest involving interdisciplinary problem solving and critical thinking skills
- Availability of enough **data** for a variety of approaches and depth of solutions (but no easily found answers)
- References identified that will be helpful for getting students started
- Submitted problem idea in the format of previous Challenge problems
- Potential to extend and enhance model using technical computing if a team chooses to do so.

Watch a video that explains the Challenge in one minute!
Go to YouTube and Search on "About MathWorks Math Modeling Challenge"

Submit your ideas: *M3Challenge.siam.org/suggest-problems*

View previous problem statements: M3Challenge.siam.org/resources/sample-problems
Contact SIAM for more information: M3Challenge@siam.org

M3Challenge.siam.org

MathWorks Math Modeling Challenge
A program of SIAM

The National Association of Secondary School Principals has placed this program on the NASSP National Advisory List of Student Contests and Activities since 2010.

A program of SIAM | Society for Industrial and Applied Mathematics

Sponsored by MathWorks

Elliptical Eyes, Eccentricity, and the Speed of Light

My older son, who enjoyed collecting shiny objects as a boy, kept a glass ball on his windowsill. The ball acted as a lens and burned a quarter-sized dip into the sill. The dip was presumably an approximation of the caustic — the envelope of the family of refracted rays.

This experience prompted the following question: What shape of solid glass would focus the light at a *point*? I realized that (i) such a shape is an ellipsoid of revolution whose major axis is parallel to the incoming beam; (ii) the eccentricity must equal the speed of light in the glass (taking the speed of light $c_{\text{air}}=1$); and (iii) the light will collect at the far focus (see Figure 1).

Two interesting limiting cases may deserve a mention:

(i) If $c_{\text{inside}} << 1$, the ellipse is a near-circle and the inside rays are near-radii.

(ii) If $c_{\text{inside}} \uparrow 1$, the far focus goes to infinity and the ellipse approaches a parabola.

Proof of focusing in Figure 1 is remarkably short. Let MN in Figure 2 be the ellipse's directrix, so that $r=ed$ defines the ellipse with eccentricity e. Differentiation with respect to the arclength parameter of point P yields

MATHEMATICAL CURIOSITIES

By Mark Levi

$$\sin\beta_1 = e\sin\alpha, \qquad (1)$$

since $r'=\sin\beta_1$ and $d'=\sin\alpha$, where $'=d/ds$ and s is the arclength parameter of P. On the other hand, $\beta_1=\beta_2$ by the optical property of the ellipse and (1) becomes

$$\sin\beta_2 = e\sin\alpha. \qquad (2)$$

This holds for an ellipse of any eccentricity. But now let us choose $e=c$, where c is the speed of light inside the ellipse; (2) then coincides with Snell's law

$$\frac{\sin\beta_2}{c} = \frac{\sin\alpha}{1}, \quad 1 = c_{\text{air}},$$

which proves that the broken line entering the far focus in Figure 2 is indeed a ray's path.

Figure 3 depicts the straight wavefront becoming an arc of a circle upon entering the ellipse.

Interestingly, fluid-filled ellipses act somewhat like parabolic mirrors. Indeed, a point source of light that is placed in a focus of Figure 2 will produce a parallel beam that exits the ellipse. The question as to what happens to the other rays (such as ones that point up from the focus) is left as a puzzle.

This discussion shows that a primitive eye—i.e., one that is filled with an optically homogeneous gel (unlike our eyes, which have lenses inside)—should be elliptical and that the retina should be placed at the far focus.

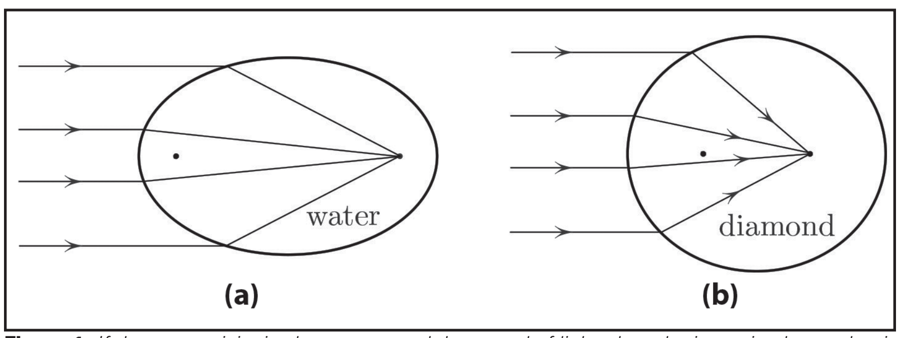


Figure 1. *If the eccentricity is chosen to equal the speed of light, then the incoming beam that is parallel to the major axis collects at the far focus.* **1a.** $e_{\text{water}}=c_{\text{water}}=.75$. **1b.** $e_{\text{diamond}}=c_{\text{diamond}}=.41$.

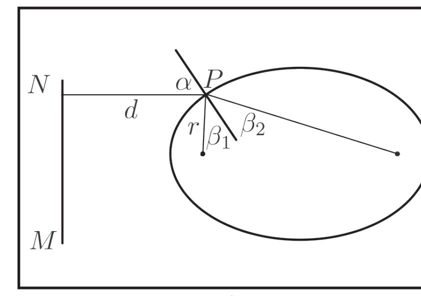


Figure 2. *For the focusing to occur in the ellipse, the eccentricity must equal the speed of light in the material (with speed of light in the air taken as 1).*

The figures in this article were provided by the author.

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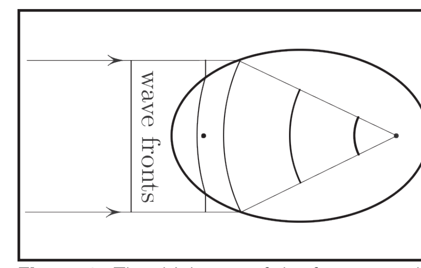


Figure 3. *The thickness of the front encodes the intensity of the light as it approaches the focus.*

SciFest All Access

Continued from page 8

awareness — all of which involve mathematics as a key component. Along with these subjects are 21st-century lifelong competencies like communication, collaboration, critical thinking, and creativity. However, educators must start introducing appropriate *contexts* and relevant *frameworks* if they want the next generation of mathematicians to build upon and develop these competencies [1].

One way to encourage students to employ mathematics in real-world contexts is via the United Nations Sustainable Development Goals[4] — a collection of 17 interlinked global goals that are designed as a "blueprint to achieve a better and more sustainable future for all" (see Figure 2). Educators can have students select a goal (i.e., Goal 2: Zero Hunger) towards which they would like to make an impact, then introduce them to the art of model development. Students should understand and define a problem statement that allows them to generate a mathematical model from a physical context. This could be an algebraic or differential equation model that describes the dynamics associated with quantities of interest (e.g., growth of a population, size of a spread, etc.) in the problem.

After developing a model, students must learn to analyze it in order to investigate and theorize appropriate behaviors with rigorous mathematical tools. This kind of analysis frequently yields important estimates or bounds that are useful for comparison purposes. Because most models that represent the real world may not admit exact solutions, educators should promote the use of numerical methods to test these complex models. The testing step often involves simulation, validation, and prediction, all of which evaluate a model's predictive capability. Working through these steps automatically helps students develop their essential habits or competencies as an integrated and sustained process. This framework can contribute to the organization of mathematical problem solving and the creation of agents of change at the student, faculty, and institutional levels [2].

References

[1] Seshaiyer, P. (2021). Novel frameworks for upskilling the mathematics education workforce. In D. Burghes & J. Hunter (Eds.), *Mathematics education for sustainable economic growth and job creation*. New York, NY: Routledge.

[2] Seshaiyer, P., & Caiseda, C. (2021). From context to habit: A new CMATH framework for undergraduate research. In *Future of undergraduate research in mathematical sciences*. Washington, DC: Mathematical Association of America. To appear.

[3] Seshaiyer, P., Mubayi, A., & MaClean, R. (2020, December 18). COVID-19: Models, mathematics and myths. *SIAM News*. Retrieved from https://sinews.siam.org/Details-Page/covid-19-models-mathematics-and-myths.

Padmanabhan Seshaiyer is a professor of mathematical sciences at George Mason University and chair of the SIAM Diversity Advisory Committee. He works in the broad area of computational mathematics, data science, biomechanics, design and systems thinking, and STEM education. Seshaiyer also serves as vice chair of the U.S. National Academies Commission on Mathematics Instruction. Earlier this year, he was officially appointed to the newly formed Virginia STEM Advisory Board to the Governor of Virginia.

[4] https://sdgs.un.org/goals

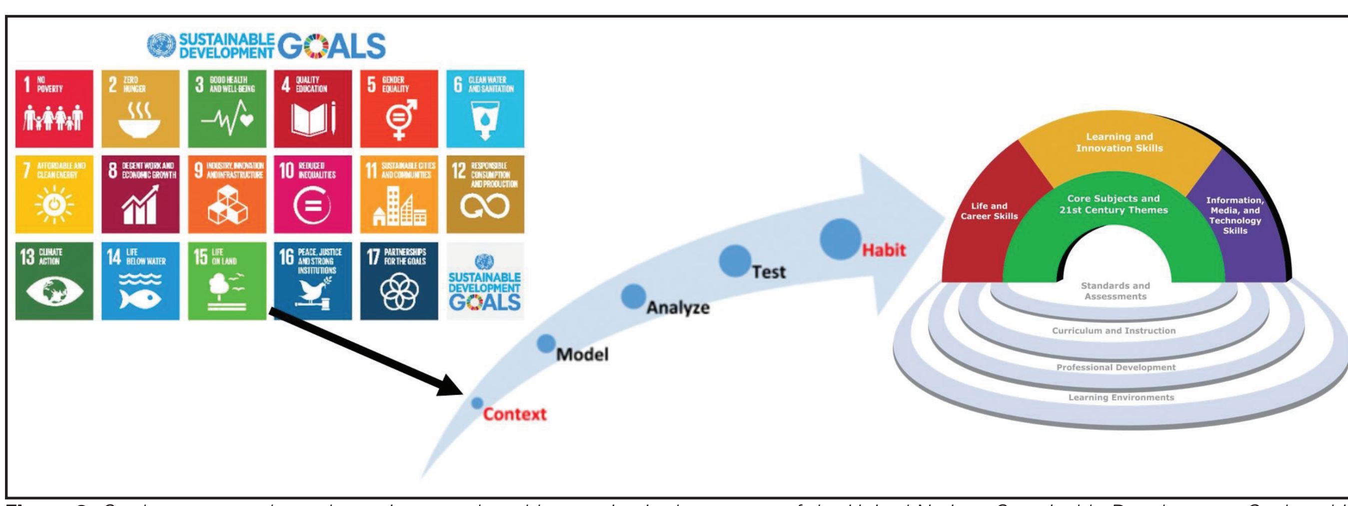


Figure 2. *Students can apply mathematics to real-world scenarios in the context of the United Nations Sustainable Development Goals, which are designed to help society achieve a more sustainable future. Figure courtesy of the author.*

Ten Memorable Planar Curves and Their Remarkable Histories

BOOK REVIEW
By Ernest Davis

Curves for the Mathematically Curious: An Anthology of the Unpredictable, Historical, Beautiful, and Romantic. *By Julian Havil. Princeton University Press, Princeton, NJ, October 2019. 280 pages, $29.95.*

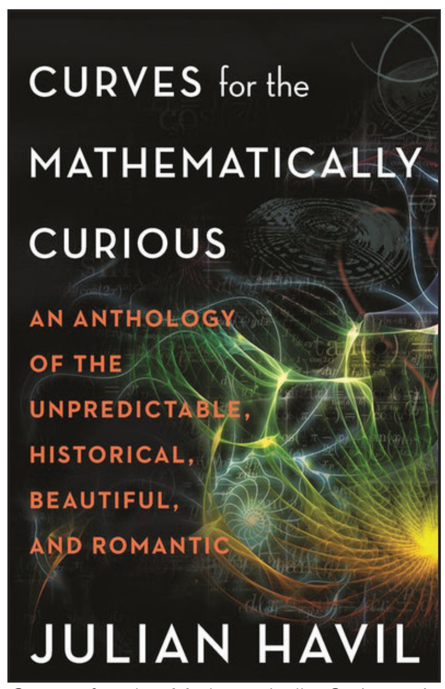
Eight years ago, Ian Stewart published *In Pursuit of the Unknown: 17 Equations That Changed the World*. The book—along with an associated wall poster that displayed the 17 equations—was very popular with a large audience of math enthusiasts. Books consisting of vignettes that pertain to one particular category date back to at least 1851, with Edward Creasy's *The Fifteen Decisive Battles of the World: From Marathon to Waterloo*. The genre has become extremely popular in recent years, as the framework provides a handy, flexible organizational structure for both the writer and reader. The field of computer science has *Nine Algorithms that Changed the Future: The Ingenious Ideas That Drive Today's Computers* by John MacCormick. General history has titles like *A History of the World in 12 Maps* by Jerry Brotton, *A History of the World in 100 Objects* by Neil MacGregor, and the Smithsonian's *History of the World in 1,000 Objects*; indeed, authors have penned histories in 100 objects about women, the Tudors, Ireland, American sports, the Third Reich, and so forth.

Julian Havil's newest book—*Curves for the Mathematically Curious: An Anthology of the Unpredictable, Historical, Beautiful, and Romantic*—follows this pattern and illuminates episodes in the history of mathematics with 10 curves: the Euler spiral, the continuous but nowhere differentiable Weierstrass function, Bezier curves (i.e., cubic splines), the rectangular hyperbola, the quadratrix of Hippias, space-filling curves, constant-width curves, the normal curve, the catenary, and elliptic curves. Havil is a retired former master at Winchester College, an elite public school in Hampshire, England.

The book contains a great deal of valuable mathematics and mathematical history. I personally found the chapter on the Euler spiral—with its beautiful equation "curvature = arc length"—particularly satisfying from a geometric viewpoint. And the backstories of the normal curve (from Abraham de Moivre to Carl Friedrich Gauss) and the space-filling curve (from Georg Cantor and Richard Dedekind to Giuseppe Peano and David Hilbert) offered fascinating historical insight.

However, readers must prepare to work; Havil does not coddle them. He immediately gets down to business in the second paragraph of chapter one: *The standard practice of expressing a curve in parametric form $x = g(t)$, $y = h(t)$ brings with it variants of formulae for its common characteristics: its slope, the area under it, its arc length, and its curvature.* This sentiment essentially sets the tone for the book as a whole.

For the most part, the mathematical toolbox upon which Havil relies is remarkably limited and mainly consists of high school geometry, algebra, trigonometry, and calculus. For instance, the chapter about the normal curve calls for a minimal amount of probability theory. Polar coordinates, vectors, geometric transformations, three-dimensional geometry, non-Euclidean geometry, and projective geometry go entirely unmentioned. Complex numbers appear only a few times in passing. Havil uses matrices to abbreviate systems of linear equations but not for any geometric purposes. However, the chapter on elliptic curves is an exception to the book's mathematic simplicity in that it involves a much broader and deeper mathematical background than the rest of the text, with significant elements of number theory and some discussion of group theory; unavoidably, this coverage is much less complete.



Curves for the Mathematically Curious: An Anthology of the Unpredictable, Historical, Beautiful, and Romantic. By Julian Havil. Courtesy of Princeton University Press.

Havil wields these mostly limited tools with great power and efficacy, if not extraordinary elegance or grace. The proof of de Moivre's formula (a step toward the normal curve), which states that

$$\frac{1}{2^n}\binom{n}{n/2} \approx 2\frac{21}{125}\left(1-\frac{1}{n}\right)^n \frac{1}{\sqrt{n-1}}$$

for even n, involves four pages of rather uninspiring symbol manipulation.

There is only one elegant proof in the book and it just gets a quick reference. At the very end of the chapter on constant-width curves, Havil quotes a theorem of Arthur Mellish. This theorem states that the following four conditions on a curve are equivalent: (i) it has constant width, (ii) it has constant diameter, (iii) all of its normals are double, and (iv) the sum of the radii of curvature at opposite points is constant. Havil then notes that "If we embrace [Hermann] Minkowski's addition of ovals, it is quite obvious; it is not difficult to show that the sum of a curve of constant width with the same curve rotated through 180° is a circle of radius that width. The result is immediate."

To a happy-go-lucky American like myself, Havil comes across as a no-nonsense British maths instructor of the old school. He might be willing to spend a few minutes after class showing students how to use cubic splines to replicate a sketch that Picasso made of his dachshund "Lump," or demonstrating how one can express any 106×17 pixelated black-and-white text in terms of the three inequalities $0 \leq x < 106$, $N \leq y < N+17$, and

$$\frac{1}{2} < \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2\right)$$

for the proper choice of N. But before indulging in such frivolities, one must work through the three-page proof that confirms that the Weierstrass function $W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n nx)$ is nowhere differentiable.

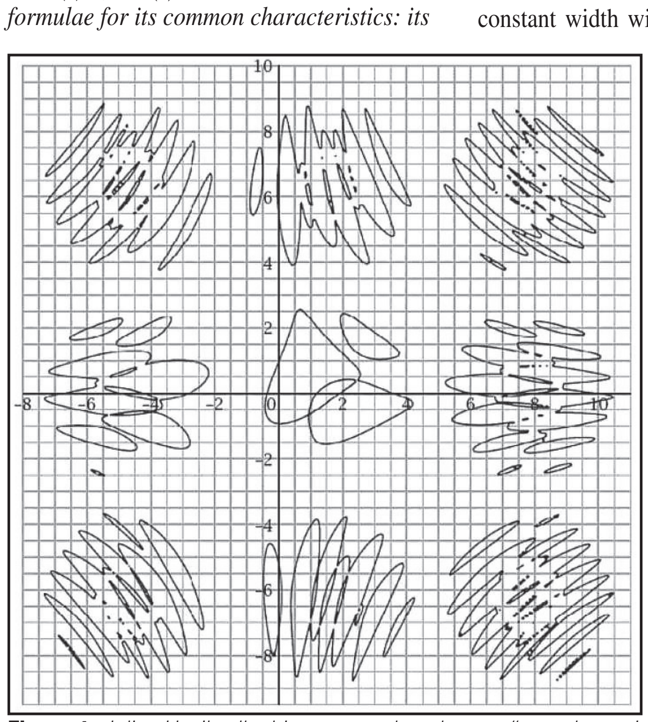


Figure 1. *Julian Havil calls this transcendental curve "a mathematical doodle." Figure courtesy of Princeton University Press.*

In fairness, Havil's choice of theorems—and methods with which to prove them—presumably reflects his historical interests instead of purely mathematical criteria. He likes to present theories as they actually developed and proofs as they were constructed at the time; of course, these are now often of more historical than mathematical interest. Thankfully, Havil uses current terminology and notation rather than the original notations.

Curves for the Mathematically Curious includes a lot of interesting history of which I was not aware. Most of it stems from the Greek/Hellenistic era and the period from Newton to 1900, though the chapter on elliptic curves also contains compelling content from medieval and 20th-century mathematics. All of this text represents substantial, careful research and analysis on Havil's part. The material sticks closely to mathematics and its applications; it does not dilly-dally with issues of biography, culture, or more general history. The "romance" in the book's subtitle presumably refers to the equation for the conventional heart shape—$x^2 + (3y/4 - \sqrt{|x|})^2 = 1$—that Havil provides at the end of the book. The only other mention of tender passion is Cantor's honeymoon in Interlaken, Switzerland in 1874; Cantor spent much of this time discussing mathematical issues with Dedekind, who was also vacationing there.

Havil raises one mystery that he neither discusses nor explains. At the very beginning of the book—before even the epigraph or table of contents—he includes the image in Figure 1: the graph of the equation $\sin(\sin x + \cos y) = \cos(\sin xy + \cos x)$. This figure is certainly not mathematically elegant and probably not mathematically important, but it intrigues me nonetheless. Some of the plot's characteristics—the square lattice arrangement of the blobs at intervals of 2π, the imperfect symmetries, the blobs' increasing complexity—are easily explainable in terms of the equation, but the precise forms are remarkable. Is there anything deeper here? What accounts for the lines of tiny closed curves (the dots are presumably all tiny closed curves) that are strung out like pearls inside the larger blobs? Do the blobs have a limiting behavior as one goes toward infinity in different directions? Is the image topologically correct, or has the plotting software introduced artifacts and flaws? Is this just a random pattern that Havil found while experimenting with plotting software, or does it have some more profound significance? Inquiring minds want to know.

All in all, readers who are interested in planar geometry and its history and are eager to satisfy their curiosity will find much to enjoy and learn in *Curves for the Mathematically Curious*.

Ernest Davis is a professor of computer science at New York University's Courant Institute of Mathematical Sciences.

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The Continued Evolution of SIAM Conferences

By Richard Moore

The 2022 ACM-SIAM Symposium on Discrete Algorithms (SODA22)[1]—scheduled for January 9-12 in Alexandria, Va.—will be SIAM's first in-person conference since the onset of COVID-19. SODA22 and its joint meetings—the SIAM Symposium on Algorithm Engineering and Experiments,[2] SIAM Symposium on Simplicity in Algorithms,[3] and SIAM Symposium on Algorithmic Principles of Computer Systems[4]—will actually be *hybrid*, with both in-person components that allow vaccinated participants to gather in Alexandria and a virtual option that enables participation from anywhere in the world.

SIAM has committed to offering all of its conferences[5] in either hybrid or virtual format through the 2022 SIAM Annual Meeting,[6] which will take place in Pittsburgh, Pa., in July along with the SIAM Conference on the Life Sciences,[7] SIAM Conference on Applied Mathematics Education,[8] and SIAM Conference on Mathematics of Planet Earth.[9] This decision, which acknowledges continuing impediments to international travel and concerns about crowded conference venues, also broadens participation and access for those who are not able or ready to attend in-person gatherings. In addition, it presents an opportunity to add to the wealth of information that stems from SIAM's forced experiment with virtual formats.

What did we learn in 2021? First, virtual conferences have real value. We know this from both the surprisingly high conference registration numbers throughout the year and positive survey responses. Nine out of the 11 virtual conferences in 2021 broke previous registration records for their conference series, and all of the attendance numbers were higher than each conference's preceding iteration (increases ranged from one to 93 percent). Not counting the 2020 SIAM Annual Meeting—where free registration meant that a number of attendees spent little to no time on the platform—the new high-water mark for registrations at a SIAM conference is 2,542 at the 2021 SIAM Conference on Computational Science and Engineering, followed closely by the 2021 SIAM Annual Meeting (co-located with the SIAM Conference on Applied and Computational Discrete Algorithms, SIAM Conference on Control and Its Applications, SIAM Conference on Discrete Mathematics, and SIAM Conference on Optimization) with 2,202 registrants. Many of these virtual registrants are based outside the U.S. (see Figures 1 and 2). Between 78 and 96 percent of survey respondents agreed or strongly agreed that the conferences' technical programs were excellent, thus demonstrating that the quality of information exchange at SIAM conferences did not suffer in a virtual modality.

Second, opinions vary widely as to what constitutes a good virtual platform. When we embarked on the exploration process for a new platform vendor in fall 2020, we included stakeholders from SIAM staff, SIAM leadership, and the organizing committees of upcoming conferences so that a variety of voices could express their priorities. We sought a single platform with which SIAM staff could become familiar and build robust practices to support, but one that had sufficient flexibility to accommodate the many different conference variations preferred by SIAM's Activity Groups and conference steering committees. The months-long process of multiple vendor calls and walkthroughs yielded the 2021 platform powered by vFairs that many readers have experienced.

Overall, the vFairs platform has rated well. With the exception of SODA21 (the first conference on the platform), between 63 and 80 percent of survey respondents indicated that their experiences with the platform were positive or very positive. We quickly phased out features that were sources of irritation—like leaderboards and gamification—and adapted our procedures to promote increased levels of interactivity during sessions, including use of Zoom's more egalitarian meeting format rather than its tiered access webinar format. We augmented virtual posters to allow for multiple media types—from the usual PDFs to video walkthroughs—and poster presenters had access to a video room in which they could engage with small groups, just as they would in person. We also appended a Gather space, complete with games and a virtual beach, to each conference to enable more spontaneous interactions. In some conferences, the Gather space was bustling with activity; in others, it was a ghost town. Fostering random conversations such as those that commence in the coffee line is a tall order for an online platform.

Finally, we learned that we can do better. The last two years have forced us to distill, adapt, and reinvent almost five decades of in-person conference experience, and the process has not been without hiccups. Readers will notice that I excluded SODA21 from my discussion of survey results on virtual platform satisfaction. This is because a woeful 38 percent of respondents expressed a positive or very positive experience with the platform. Since SODA21 took place just a few short months after we signed our contract with vFairs, we needed to move quickly to set up the platform and communicate with all participants. We also wanted to accommodate the organizers' request for pre-recorded talks from all contributed speakers, who then had to record and upload their talks during a very short window over the holidays. Registrants consequently had little time to access the recorded talks before the live sessions; survey scores suffered as a result, and we realized that adding functionality does not translate to additional value without the necessary time and planning to bring users along for the ride.

What does all of this mean for hybrid conferences in 2022? Because adaptation is a healthy exercise, we are moving to a different vendor: Pathable. This new platform is much more mobile-friendly but still allows browser access, adapts automatically to local time zones, and interacts seamlessly with calendar apps — all while offering easy access to Zoom meetings that will now be connected to onsite conference rooms. Pathable will therefore serve as both a conference scheduling app for in-person attendees and a virtual conference platform for online participants. Speakers and attendees will be able to partake in sessions from either the venue or their own homes. The 2022 SIAM Conference on Analysis of Partial Differential Equations[10] and SIAM Conference on Imaging Science[11]—both originally scheduled for Germany in March—will be fully virtual and take place on SIAM's vFairs platform.

As for our plan for future meetings once COVID-19 is safely in the rearview mirror, I have learned to avoid making lofty predictions. I hope that our newfound experience with hybrid conferences allows us to continue increasing global access to SIAM meetings and networking opportunities. For now, I'm just looking forward to seeing friends and colleagues in person again.

Richard Moore is the Director of Programs and Services at SIAM.

[1] https://www.siam.org/conferences/cm/conference/soda22
[2] https://www.siam.org/conferences/cm/conference/alenex22
[3] https://www.siam.org/conferences/cm/conference/sosa22
[4] https://www.siam.org/conferences/cm/conference/apocs22
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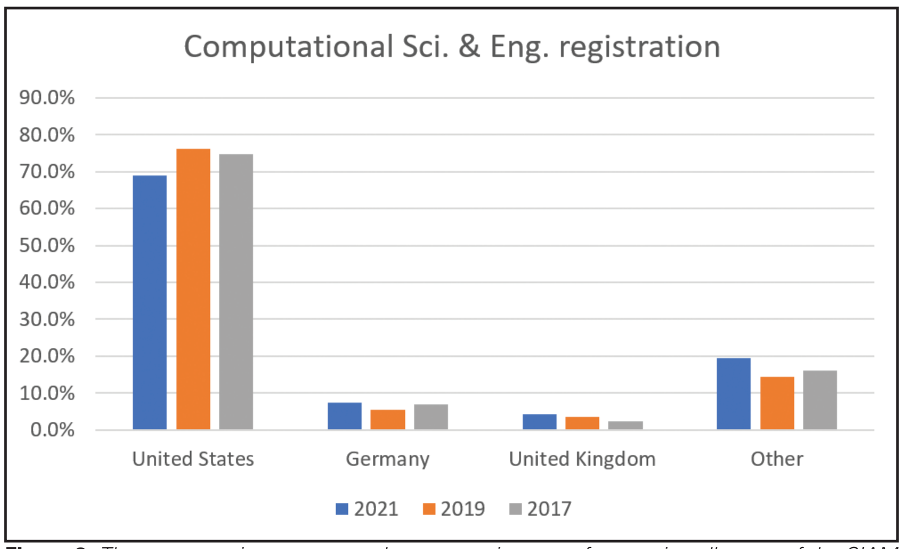

Figure 2. *The top countries represented among registrants of recent installments of the SIAM Conference on Computational Science and Engineering, held in person in 2017 and 2019 and virtually in 2021. Figure courtesy of the author.*

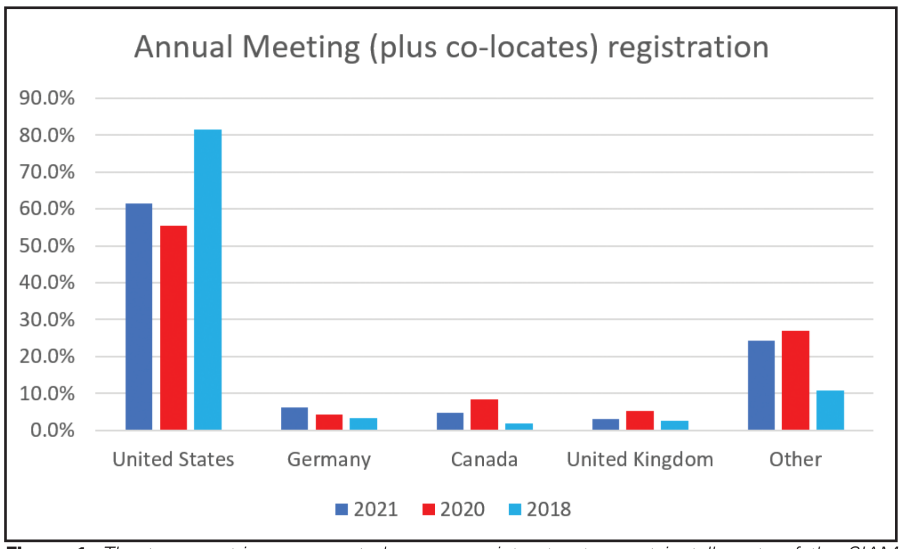

Figure 1. *The top countries represented among registrants at recent installments of the SIAM Annual Meeting, held in person in 2018 and virtually in 2020 and 2021. Figure courtesy of the author.*