Optimizing Energy Functional in Wave and Heat Equations with Initial Conditions in a Class of Rearrangements

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Abstract. This paper aims to maximize the energy functionals of the Cauchy problem for one-dimensional wave and heat equations through a rearrangement class of initial conditions. The energy functional is defined to be the classical physical energy on a restricted interval. The corresponding result of wave equations is considered separately for three cases and that of heat equations can be generalized to higher dimensions. Moreover, uniqueness of the solution and stability problems are studied for heat equations.

1. Introduction.

Wave equations model the vibration and wave propagation phenomena [8]. In particular, the classical one-dimensional Cauchy problem for the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

governs the motion of a stretched string for some initial displacement function $f(x)$ and initial velocity $g(x)$ where $c > 0$ is the wave speed, $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. The corresponding physical energy functional is defined as

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left({u_t}^2 + c^2 u_x^2 \right) dx \tag{1.2}$$

and we know $E(t)$ is conserved, i.e., $E'(t) = 0$ under certain conditions (See Section 4.3 in [6]). For convenience, we choose to drop the normalizing constant $\frac{1}{2}$ and replace the infinite integral with a more comfortable integration over a bounded interval $[-L, L]$. In other words, we let L be any positive constant and recast our energy functional as follows

$$E_L(f,g,t) = \int_{-L}^{L} \left[\left(\frac{\partial u_{f,g}}{\partial t} \right)^2 + c^2 \left(\frac{\partial u_{f,g}}{\partial x} \right)^2 \right] dx \tag{1.3}$$

where $u_{f,g}$ is the solution of (1.1) given by the d'Alembert's formula (See Section 4.2 in [6]):

$$u_{f,g}(x,t) = \frac{1}{2}[f(x+ct) - f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy. \tag{1.4}$$

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Note that here we make the sub-index $u_{f,g}$ to emphasize the dependence of initial conditions f and g. Roughly speaking, our aim is to find some appropriate initial conditions f and g that maximize (1.3). In what follows, we consider the admissible set for possible initial conditions to be the class of rearrangements:

$$\mathcal{R}(\eta) := \{h : \mu_h(s) = \mu_\eta(s) \quad \text{for all } s \in \mathbb{R}\}, \tag{1.5}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a fixed measurable function, $\mu_h(t) = |\{x : h(x) > t\}|$ and $|A|$ denotes the measure of a Lebesgue measurable set A. μ_h is also called the distribution function of h. So we consider our problem in the following generalized form:

Problem 1.1. *Fix non-negative $g_0 \in L^2(\mathbb{R})$ and $f_0 \in H^1(\mathbb{R})$ where $H^1(\mathbb{R})$ denotes the Sobolev space such that the function and its distributional derivative are both in $L^2(\mathbb{R})$. Find two functions $g \in \mathcal{R}(g_0)$ and $f \in \mathcal{R}(f_0)$ (if possible) such that $E_L(f, g, t)$ attains the maximum.*

If f_0 is non-trivial and $|\{f_0 > 0\}| \le 2L$, it shall be proved that

$$\sup_{\substack{f\in\mathcal{R}(f_0)\\ g\in\mathcal{R}(g_0)}} E_L(f, g, t) = \infty.$$

For the special case $f_0 = 0$ and $ct \le L$,

$$\sup_{\substack{f\in\mathcal{R}(f_0)\\ g\in\mathcal{R}(g_0)}} E_L(f, g, t) = E_L(g^\star, t)$$

where we applied an abuse of notation as $f = 0$ is fixed. Note that $g^\star$ denotes the Schwarz symmetrization or symmetric decreasing rearrangement of g, which we will define in Section 2.

Heat equations describe the process of heat conduction in isotropic bodies [8]. In particular, the classical one-dimensional Cauchy problem for the heat equation

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x, 0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

models the distribution of temperature in a rod where $k > 0$ is the diffusion constant and $h \in L^1(\mathbb{R})$. The corresponding physical energy is

$$E(t) = \int_{-\infty}^{\infty} u(x, t)dx \tag{1.7}$$

and $E(t)$ is conserved under certain conditions (See Section 5.3 in [6]). Similarly, as discussed for wave equations, our new energy functional is defined as

$$E_L(h, t) = \int_{-L}^{L} u_h(x, t)dx \tag{1.8}$$

where u_h is the particular (or tailor-made) solution to (1.6) given by:

$$u_h(x, t) = \int_{-\infty}^{\infty} \Phi(x - y, t)h(y)dy \tag{1.9}$$

with

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

to be the fundamental solution of the heat equation. In what follows, our problem is formulated as:

Problem 1.2. *Fix a non-negative $h_0 \in L^1(\mathbb{R})$. Find one $h \in \mathcal{R}(h_0)$ (if possible) such that $E_L(h,t)$ attains the maximum. In other words, we intend to solve*

$$\sup_{h\in\mathcal{R}(h_0)} E_L(h,t). \tag{1.10}$$

Different from the results in wave equations, we have

$$\sup_{h\in\mathcal{R}(h_0)} E_L(h,t) = E_L(h^\star,t).$$

Additionally, uniqueness and stability problems will also be addressed.

2. Schwarz Symmetrization. In this section, we firstly introduce the notion of Schwarz symmetrization and several related useful properties and inequalities.

If A is a measurable set of finite measure in $\mathbb{R}^n$, we define $A^\star$, the *symmetric rearrangement of the set* A, to be the open ball centered at the origin whose volume is the same as that of A. Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be a measurable function vanishing at infinity, i.e., $|\{x : f(x) > t\}|$ is finite for all $t > 0$. For $E \subseteq \mathbb{R}^n$, the notation $\chi_E(x)$ stands for the characteristic function of E, i.e., $\chi_E(x) = 1$ for $x \in E$ while $\chi_E(x) = 0$ for $x \notin E$. $f^\star$ is defined as the "unique" symmetric decreasing rearrangement (also called the Schwarz symmetrization) of f:

$$f^\star(x) := \int_0^\infty \chi_{\{f>t\}^\star}(x)dt,$$

where we use $\{f > t\}$ to denote the upper level set $\{x : f(x) > t\}$. Some of the important properties of $f^\star$ that we shall use later on are listed as follows:

1. $f^\star$ is radially symmetric and non-increasing, i.e.,

$$f^\star(x) \geq f^\star(y) \quad \text{if } |x| \leq |y|$$

and $f^\star(x) = f^\star(y)$ if $|x| = |y|$. Incidentally, we say that $f^\star$ is *strictly symmetric decreasing* if $f^\star(x) > f^\star(y)$ if $|x| < |y|$.

2. If $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, then

$$(\Psi \circ f)^\star = \Psi \circ f^\star. \tag{2.1}$$

3. If f, $g \in L^p(\mathbb{R}^n)$, where $1 \leq p \leq \infty$. Then

$$\|f^\star - g^\star\|_p \leq \|f - g\|_p. \tag{2.2}$$

Finally, we state a well-known result:

Proposition 2.1. *Fix a non-negative $g_0 \in L^2(\mathbb{R})$. If $g \in \mathcal{R}(g_0)$, then $g \in L^2(\mathbb{R})$ and furthermore we have $\|g\|_2 = \|g_0\|_2$, where $\|\cdot\|_2$ denotes the L^2 norm.*

Proof. This is a direct consequence of layer cake representation and Fubini Theorem, see Lemma 2.1 in [3] or the point (iv) in Section 3.3 of [5]. ■

For more details of discussion, see Chapter 3 in [5] and Chapter 1 in [4]. On the other hand, for a rich survey on the development of the rearrangement theory, we refer to Talenti's article [7].

3. Energy Optimization in One-Dimensional Wave Equations with Initial Velocity. We firstly consider our Problem 1.1 when $f_0 = 0$. In this section, we use u_g to denote the d'Alembert solution (1.4) for (1.1) with $f = 0$. This time our energy functional (1.3) becomes

$$E_L(g,t) = \int_{-L}^{L} \left[(u_g)_t^2 + c^2(u_g)_x^2\right] dx. \tag{3.1}$$

Some basic properties of our solution are initially examined.

Proposition 3.1. *Let v be the solution of the following Cauchy problem:*

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = g^\star(x), & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Then $v = v^\star$.

Proof. By d'Alembert's formula (1.4), we have $v(x,t) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g^\star(y)dy$. Firstly, v is even. Indeed,

$$v(-x,t) = \frac{1}{2c}\int_{-x-ct}^{-x+ct} g^\star(y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct} g^\star(-y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct} g^\star(y)dy = v(x,t),$$

where we used the fact that $g^\star$ is even in the third equality. Then we show v is decreasing in x for $x > 0$. In fact, when $0 < x < ct$,

$$\frac{\partial v}{\partial x} = \frac{1}{2c}[g^\star(x+ct) - g^\star(x-ct)] = \frac{1}{2c}[g^\star(ct+x) - g^\star(ct-x)] \le 0$$

and when $x \ge ct$,

$$\frac{\partial v}{\partial x} = \frac{1}{2c}[g^\star(x+ct) - g^\star(x-ct)] \le 0.$$

So then by the uniqueness of symmetric decreasing rearrangement, we conclude that $v = v^\star$. ■

Now we can show the existence of $\sup_{g\in\mathcal{R}(g_0)} E_L(g,t)$.

Proposition 3.2. *Fix $L > 0$. Then $\sup_{g\in\mathcal{R}(g_0)} E_L(g,t)$ is finite.*

Proof. By direct calculation, we have

$$(u_g)_t = \frac{1}{2}[g(x+ct)+g(x-ct)] \quad \text{and} \quad (u_g)_x = \frac{1}{2c}[g(x+ct)-g(x-ct)].$$

Hence, $E_L(g,t)$ is computed as

$$\begin{aligned} E_L(g,t) &= \int_{-L}^{L} [(u_g)_t^2 + c^2(u_g)_x^2]dx \\ &= \frac{1}{2}\int_{-L}^{L} \left[g^2(x+ct)+g^2(x-ct)\right] dx. \end{aligned} \tag{3.3}$$

Using Proposition 2.1, we have

$$\begin{aligned} E_L(g,t) &\leq \frac{1}{2}\int_{-\infty}^{\infty} g^2(x+ct)dx + \frac{1}{2}\int_{-\infty}^{\infty} g^2(x-ct)dx \\ &= \int_{-\infty}^{\infty} g^2(x)dx = \int_{-\infty}^{\infty} g_0^2(x)dx < \infty. \end{aligned}$$

∎

Now, we can state one of the main results we obtain for wave equations:

Theorem 3.3. *Fix $L>0$. If $ct \leq L$, then $E_L(g,t) \leq E_L(g^\star,t)$, i.e., $\sup_{g\in\mathcal{R}(g_0)} E_L(g,t) = E_L(g^\star,t)$.*

Proof. Using (3.3) and making a change of variable, we have

$$\begin{aligned} E_L(g,t) &= \frac{1}{2}\int_{-L}^{L} \left[g^2(x+ct)+g^2(x-ct)\right] dx \\ &= \frac{1}{2}\int_{-L+ct}^{L+ct} g^2(x)dx + \frac{1}{2}\int_{-L-ct}^{L-ct} g^2(x)dx \\ &= \frac{1}{2}\int_{-L-ct}^{L+ct} g^2(x)dx + \frac{1}{2}\int_{-L+ct}^{L-ct} g^2(x)dx \end{aligned} \tag{3.4}$$

when $ct \leq L$. Then by Hardy-Littlewood inequality (See Theorem 3.4 in [5]) we have

$$\begin{aligned} \int_{-L-ct}^{L+ct} g^2(x)dx &= \int_{-\infty}^{\infty} g^2(x)\chi_{[-L-ct,L+ct]}(x)dx \\ &\leq \int_{-\infty}^{\infty} (g^2)^\star(x)\chi^\star_{[-L-ct,L+ct]}(x)dx = \int_{-L-ct}^{L+ct} (g^\star)^2(x)dx, \end{aligned}$$

where $(g^2)^\star(x) = (g^\star)^2(x)$ in the last equality is due to (2.1). Similarly, we obtain

$$\int_{-L+ct}^{L-ct} g^2(x)dx \leq \int_{-L+ct}^{L-ct} (g^\star)^2(x)dx.$$

Combining these two inequalities, (3.4) becomes

$$\begin{aligned}E_L(g,t) &\leq \frac{1}{2}\left(\int_{-L-ct}^{L+ct}(g^\star)^2(x)dx + \int_{-L+ct}^{L-ct}(g^\star)^2(x)dx\right)\\ &= \frac{1}{2}\int_{-L}^{L}[(g^\star)^2(x+ct)+(g^\star)^2(x-ct)]dx\\ &= E_L(g^\star,t).\end{aligned}$$

■

However, the proof of Theorem 3.3 does not work for the case when $ct > L$. This is because our energy functional becomes

$$E_L(g,t) = \frac{1}{2}\int_{-L-ct}^{L+ct} g^2(x)dx - \frac{1}{2}\int_{L-ct}^{-L+ct} g^2(x)dx,$$

and the sign in front of the second term is negative. Indeed, we can give a counterexample that a similar result as for Theorem 3.3 is not correct, i.e., the symmetric decreasing rearrangement of g does not yield the maximal energy $E_L(g,t)$ in general.

Example 3.4. *Consider the following Cauchy problem:*

$$\begin{cases} u_{tt} = u_{xx}, & x\in\mathbb{R},\ t>0,\\ u(x,0)=0, & x\in\mathbb{R},\\ u_t(x,0)=g(x), & x\in\mathbb{R},\end{cases} \tag{3.5}$$

where

$$g(x) = \begin{cases}\sqrt{x}, & \text{if } x\in[0,1],\\ \sqrt{2-x}, & \text{if } x\in[1,2],\\ 0, & \text{if } x\notin[0,2].\end{cases}$$

Let $L=1$ and $1<t<2$. Observing that $c=1$, $E_L(g,t)$ (see Figure 1) is computed as

$$E_L(g,t) = \frac{1}{2}\left(\int_{-1-t}^{1+t} g^2(x)dx - \int_{1-t}^{-1+t} g^2(x)dx\right) = \frac{1}{2}\int_{-1+t}^{2} g^2(x)dx = \frac{1}{2}\left[1-\frac{1}{2}(t-1)^2\right].$$

Now

$$g^\star(x) = \begin{cases}\sqrt{1-x}, & \text{if } x\in[0,1],\\ \sqrt{x+1}, & \text{if } x\in[-1,0],\\ 0, & \text{if } x\notin[-1,1],\end{cases}$$

and $E_L(g^\star,t)$ (see Figure 2) is similarly computed to be

$$E_L(g^\star,t) = \frac{1}{2}\left(\int_{-1}^{1-t}(g^\star)^2(x)dx + \int_{-1+t}^{1}(g^\star)^2(x)dx\right) = \frac{1}{2}(2-t)^2.$$

Thus, we have $E_L(g^\star,t) < E_L(g,t)$ when $1<t<2$, which is a desired counterexample.

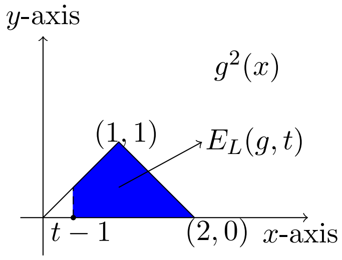

Figure 1. $E_L(g,t)$

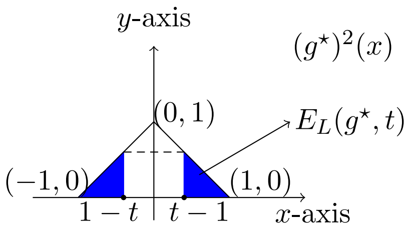

Figure 2. $E_L(g^\star,t)$

4. Energy Optimization in One-Dimensional Wave Equations with Initial Displacement. Now we consider the case when $g_0 = 0$ and f_0 is non-trivial. In this section, we use u_f to denote the d'Alembert solution of (1.1) when $g = 0$. This time our energy functional becomes

$$E_L(f,t) = \int_{-L}^{L} \left[(u_f)_t^2 + c^2(u_f)_x^2\right] dx. \tag{4.1}$$

We aim to study

$$\sup_{f\in\mathcal{R}(f_0)} E_L(f,t). \tag{4.2}$$

Surprisingly, different from the case with only initial velocity, the energy functional goes to infinity when we consider certain rearrangement sequences of f_0.

Lemma 4.1. *Fix $L > 0$, and suppose $|\{f_0 > 0\}| \le 2L$. Then $\sup_{f\in\mathcal{R}(f_0)} E_L(f,t) = \infty$ when $ct \le L$.*

Proof. Performing a similar calculation as in (3.4), we have

$$\begin{aligned} E_L(f,t) &= \frac{c^2}{2}\int_{-L}^{L}(f')^2(x+ct)dx + \frac{c^2}{2}\int_{-L}^{L}(f')^2(x-ct)dx \\ &= \frac{c^2}{2}\int_{-L+ct}^{L+ct}(f')^2(x)dx + \frac{c^2}{2}\int_{-L-ct}^{L-ct}(f')^2(x)dx \\ &= \frac{c^2}{2}\int_{-L-ct}^{L+ct}(f')^2(x)dx + \frac{c^2}{2}\int_{-L+ct}^{L-ct}(f')^2(x)dx. \end{aligned} \tag{4.3}$$

Set $f_1 = (f_0)^\star$ and note that the support of f_1 is in $[-L, L]$. Then we observe that $f_1 \in H^1(\mathbb{R})$ by Pólya-Szegö inequality (See Theorem 3.20 in [1]). On the other hand, we have $\int_{-L}^{L}(f_1')^2(x)dx > 0$. Indeed, if $\int_{-L}^{L}(f_1')^2(x)dx = 0$, then f_1 equals to some constant a.e. on $[-L, L]$ (See Lemma 8.1 in [2]). Since $|\{f_1 > 0\}| \le 2L$ and f_1 is constant, by Theorem 8.2 in [2], we have $f_1 = 0$ a.e. on $\mathbb{R}$ contradicting the fact that f_0 is non-trivial.

Now define $f_2 : \mathbb{R} \to \mathbb{R}$ to be (see Figure 3):

$$f_2(x) = \begin{cases} f_1(2x - L), & x \in [0, L], \\ f_1(2x + L), & x \in [-L, 0], \\ 0, & x \in \mathbb{R} \backslash [-L, L]. \end{cases}$$

For any $t \geq 0$, as f_1 is continuous (replace it by its continuous representation if necessary, see Theorem 8.2 and Remark 5 in Page 204 of [2]) and symmetric decreasing, we have $\{f_1 > t\} = (-s, s)$ for some $s \in [0, L]$. By our definition of f_2, one easily sees that $\{f_2 > t\} = \left(\frac{-L}{2} - \frac{s}{2}, \frac{-L}{2} + \frac{s}{2}\right) \cup \left(\frac{L}{2} - \frac{s}{2}, \frac{L}{2} + \frac{s}{2}\right)$. So $|\{f_2 > t\}| = |\{f_1 > t\}|$ and $f_2 \in \mathcal{R}(f_0)$. Moreover, we have

$$\begin{aligned} \int_{-L}^{L} (f_2')^2(x)dx &= \int_0^L \left(\frac{d}{dx} f_1(2x - L)\right)^2 dx + \int_{-L}^0 \left(\frac{d}{dx} f_1(2x + L)\right)^2 dx \\ &= 2\int_{-L}^{L} \left(\frac{df_1}{dy}(y)\right)^2 dy + 2\int_{-L}^{L} \left(\frac{df_1}{dy}(y)\right)^2 dy = 4\int_{-L}^{L} (f_1')^2(x)dx. \end{aligned} \tag{4.4}$$

Continuing this process inductively, we obtain a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{R}(f_0)$ and

$$\int_{-L}^{L} (f_n')^2(x)dx = 4^{n-1} \int_{-L}^{L} (f_1')^2(x)dx. \tag{4.5}$$

When $ct \leq L$, by (4.5), (4.3) becomes

$$\begin{aligned} E_L(f_n, t) &= \frac{c^2}{2} \int_{-L-ct}^{L+ct} (f_n')^2(x)dx + \frac{c^2}{2} \int_{-L+ct}^{L-ct} (f_n')^2(x)dx \\ &\geq \frac{c^2}{2} \int_{-L}^{L} (f_n')^2(x)dx \\ &= \frac{c^2 4^{n-1}}{2} \int_{-L}^{L} (f_1')^2(x)dx. \end{aligned}$$

Recalling that $\int_{-L}^{L} (f_1')^2(x)dx > 0$, this implies $\sup_{n\in\mathbb{N}} E_L(f_n, t) = \infty$. Consequently, we have $\sup_{f\in\mathcal{R}(f_0)} E_L(f, t) = \infty$ as desired. ∎

Actually, the above lemma is also correct for $ct > L$, which is different f rom t he c ase in Theorem 3.3 and we conclude it as follow:

Theorem 4.2. *Fix $L > 0$, and suppose $|\{f_0 > 0\}| \leq 2L$. Then $\sup_{f\in\mathcal{R}(f_0)} E_L(f, t) = \infty$.*

Proof. The case when $ct \leq L$ follows directly from Lemma 4.1. For the case when $ct > L$, we just need to modify our f_n. Indeed, define $(\bar{f}_n)$ to be

$$\bar{f}_n(x) = (f_n)^\star(x - ct) \quad x \in \mathbb{R}.$$

This is just the translation of f_n defined in Lemma 4.1 (see Figure 4) and note that $\operatorname{supp} \bar{f}_n = [-L + ct, L + ct]$. Clearly, $\bar{f}_n \in \mathcal{R}(f_n) = \mathcal{R}(f_0)$. Now, a similar calculation as in (4.3) shows that

$$\begin{aligned}E_L(\bar{f}_n,t) &= \frac{c^2}{2}\int_{-L+ct}^{L+ct}(\bar{f}_n')^2(x)dx + \frac{c^2}{2}\int_{-L-ct}^{L-ct}(\bar{f}_n')^2(x)dx \\ &= \frac{c^2}{2}\int_{-L+ct}^{L+ct}(\bar{f}_n')^2(x)dx \\ &= \frac{c^2}{2}\int_{-L}^{L}(f_n')^2(x)dx = \frac{c^2 4^{n-1}}{2}\int_{-L}^{L}(f_1')^2(x)dx\end{aligned}$$

where we used the facts $-L+ct > L-ct$ and $\operatorname{supp}\bar{f}_n = [-L+ct, L+ct]$ in the second equality, and (4.5) in the last equality. Then performing a similar argument as in Lemma 4.1, the proof is complete. ■

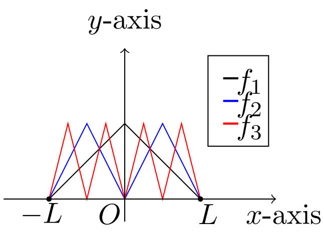

Figure 3. f_n

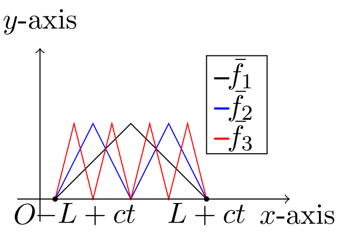

Figure 4. $\bar{f}_n$

Remark 4.3. Comparing (4.3) with (3.4), one can see that to maximize the energy functional with only initial displacement (i.e., $g = 0$) is equivalent to maximize the energy of the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = cf'(x), & x \in \mathbb{R}. \end{cases}$$

This can be interpreted in physics as: a "sharper" initial displacement yields a larger velocity, since this time there are more potential conserved in the string.

5. General Energy Optimization in One-Dimensional Wave Equation. Now we are in the position to solve Problem 1.1. Recalling $u_{f,g}$ is the solution of (1.1) and $E_L(f,g,t)$ is defined in (1.3), we have

Theorem 5.1. *Suppose f_0 is non-trivial and $|\{f_0 > 0\}| \le 2L$. Then $\sup_{\substack{f\in\mathcal{R}(f_0)\\ g\in\mathcal{R}(g_0)}} E_L(f,g,t) = \infty$.*

Proof. By d'Alembert's formula, we have

(5.1)
$$\begin{aligned}E_L(f,g,t) = \frac{1}{2}\int_{-L}^{L}\left[g^2(x+ct)+g^2(x-ct)\right]dx + \frac{c^2}{2}\int_{-L}^{L}\left[(f')^2(x+ct)+(f')^2(x-ct)\right]dx \\ + c\int_{-L}^{L}\left[f'(x+ct)g(x+ct) - f'(x-ct)g(x-ct)\right]dx.\end{aligned}$$

By Cauchy-Schwarz inequality, Young's inequality and Proposition 2.1, we have for any $\varepsilon > 0$,

$$\begin{aligned}\left|\int_{-L}^{L} f'(x+ct)g(x+ct)dx\right| &\leq \left(\int_{-L}^{L}(f')^2(x+ct)dx\right)^{\frac{1}{2}}\left(\int_{-L}^{L} g^2(x+ct)dx\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2\varepsilon}\int_{-L}^{L}(f')^2(x+ct)dx+\frac{\varepsilon}{2}\int_{-L}^{L} g^2(x+ct)dx\\ &\leq \frac{1}{2\varepsilon}\int_{-L}^{L}(f')^2(x+ct)dx+\frac{\varepsilon}{2}\left\|g_0\right\|_2^2,\end{aligned}$$

which means

$$\int_{-L}^{L} f'(x+ct)g(x+ct)dx \geq -\frac{1}{2\varepsilon}\int_{-L}^{L}(f')^2(x+ct)dx-\frac{\varepsilon}{2}\left\|g_0\right\|_2^2. \tag{5.2}$$

Similarly, we obtain

$$\begin{aligned}\int_{-L}^{L} f'(x-ct)g(x-ct)dx &\leq \left|\int_{-L}^{L} f'(x-ct)g(x-ct)dx\right|\\ &\leq \frac{1}{2\varepsilon}\int_{-L}^{L}(f')^2(x-ct)dx+\frac{\varepsilon}{2}\int_{-L}^{L} g^2(x-ct)dx\\ &\leq \frac{1}{2\varepsilon}\int_{-L}^{L}(f')^2(x-ct)dx+\frac{\varepsilon}{2}\left\|g_0\right\|_2^2.\end{aligned} \tag{5.3}$$

Using (5.2) and (5.3), (5.1) becomes

$$\begin{aligned}&E_L(f,g,t)\\ &\geq \frac{1}{2}\int_{-L}^{L}\left[g^2(x+ct)+g^2(x-ct)\right]dx+\frac{c^2}{2}\int_{-L}^{L}\left[(f')^2(x+ct)+(f')^2(x-ct)\right]dx\\ &\qquad -\frac{c}{2\varepsilon}\int_{-L}^{L}(f')^2(x+ct)dx-\frac{\varepsilon c}{2}\left\|g_0\right\|_2^2-\frac{c}{2\varepsilon}\int_{-L}^{L}(f')^2(x-ct)dx-\frac{\varepsilon c}{2}\left\|g_0\right\|_2^2\\ &\geq \left(\frac{c^2}{2}-\frac{c}{2\varepsilon}\right)\int_{-L}^{L}[(f')^2(x+ct)+(f')^2(x-ct)]dx-\varepsilon c\left\|g_0\right\|_2^2.\end{aligned}$$

Now letting $\varepsilon = \frac{2}{c}$, it follows that

$$\begin{aligned}E_L(f,g,t) &\geq \frac{c^2}{4}\int_{-L}^{L}\left[(f')^2(x+ct)+(f')^2(x-ct)\right]dx-2\left\|g_0\right\|_2^2\\ &\geq \frac{c^2}{4}\int_{-L}^{L}\left[(f')^2(x+ct)+(f')^2(x-ct)\right]dx-2\left\|g_0\right\|_2^2=\frac{1}{2}E_L(f,0,t)-2\left\|g_0\right\|_2^2\\ &=\frac{1}{2}E_L(f,t)-2\left\|g_0\right\|_2^2.\end{aligned}$$

Thus $\sup_{\substack{f\in\mathcal{R}(f_0)\\ g\in\mathcal{R}(g_0)}} E_L(f,g,t)=\infty$ by Theorem 4.2. ■

6. Energy Optimization in One-Dimensional Heat Equation. In this section, we consider Problem 1.2. Recall u_h is the solution of (1.6) and it is given by (1.9). Then, we have

Theorem 6.1. *Fix $L > 0$. Then $E_L(h,t) \le E_L(h^\star,t)$, i.e., $\sup_{h\in\mathcal{R}(h_0)} E_L(h,t) = E_L(h^\star,t)$. Moreover, $E_L(h,t) = E_L(h^\star,t)$ if and only if $h = h^\star$.*

Proof. One easily sees that $\Phi(x,t) = \Phi^\star(x,t)$ where the star operation is taken on x. Using Riesz's rearrangement inequality (Theorem 3.7 in [5]), we have

$$\begin{aligned}
E_L(h,t) &= \int_{-L}^{L}\int_{-\infty}^{\infty}\Phi(x-y,t)h(y)dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{[-L,L]}(x)\Phi(x-y,t)h(y)dydx \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi^\star_{[-L,L]}(x)\Phi^\star(x-y,t)h^\star(y)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{[-L,L]}(x)\Phi(x-y,t)h^\star(y)dydx \\
&= \int_{-L}^{L}\int_{-\infty}^{\infty}\Phi(x-y,t)h^\star(y)dydx = E_L(h^\star,t).
\end{aligned}$$

When $E_L(h,t) = E_L(h^\star,t)$, Riesz inequality takes the equality. Since the fundamental solution $\Phi(x,t)$ is strictly symmetric decreasing in x, by strict rearrangement inequality (Theorem 3.9 in [5]), we obtain $h(x) = h^\star(x)$. This completes the proof. ∎

Remark 6.2. We can also prove Theorem 6.1 by using Hardy-Littlewood inequality instead of Riesz inequality, and we include this alternative proof here to show the powerfulness of rearrangement theory. Setting the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$, the energy can be transformed into the following form

$$E_L(h,t) = \int_{-\infty}^{\infty}\frac{1}{2}\left[\operatorname{erf}\left(\frac{y+L}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{y-L}{2\sqrt{kt}}\right)\right]h(y)dy.$$

Defining $\zeta_L(y,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{y+L}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{y-L}{2\sqrt{kt}}\right)\right]$, it is easy to see ζ_L is positive and strictly symmetric decreasing in y. By Hardy-Littlewood inequality, see Theorem 3.4 in [5], we deduce

$$E_L(h,t) = \int_{-\infty}^{\infty}\zeta_L(y,t)h(y)dy \le \int_{-\infty}^{\infty}\zeta_L^\star(y,t)h^\star(y)dy = \int_{-\infty}^{\infty}\zeta_L(y,t)h^\star(y)dy = E_L(h^\star,t).$$

Moreover, if $E_L(h,t) = E_L(h^\star,t)$, then Hardy-Littlewood inequality takes the equality. As ζ_L is strictly symmetric decreasing, we can use the last assertion of Theorem 3.4 in [5] to show $h^\star$ is the unique maximizer.

We can also prove the stability of the problem.

Proposition 6.3. *Given $\varepsilon > 0$ and fixed $L > 0$. Let h_1, h_2 be two non-negative functions in $L^1(\mathbb{R})$ satisfying $\|h_1 - h_2\|_1 \le \varepsilon$, then the distance between two optimal energy values is less than ε, i.e.,*

$$|E_L(h_1^\star,t) - E_L(h_2^\star,t)| \le \varepsilon.$$

Proof. By (2.2), we have $\|h_1^\star - h_2^\star\|_1 \le \|h_1 - h_2\|_1 \le \varepsilon$. Then using Fubini theorem, we have

$$\begin{aligned}
|E_L(h_1^\star,t) - E_L(h_2^\star,t)| &= \left|\int_{-L}^{L}\int_{-\infty}^{\infty}\Phi(x-y,t)h_1^\star(y)dydx - \int_{-L}^{L}\int_{-\infty}^{\infty}\Phi(x-y,t)h_2^\star(x)dydx\right| \\
&= \left|\int_{-L}^{L}\int_{-\infty}^{\infty}\Phi(x-y,t)(h_1^\star - h_2^\star)(y)dydx\right| \\
&= \left|\int_{-\infty}^{\infty}\int_{-L}^{L}\Phi(x-y,t)(h_1^\star - h_2^\star)(y)dxdy\right| \\
&\le \int_{-\infty}^{\infty}\left|\int_{-L}^{L}\Phi(x-y,t)dx\right| |(h_1^\star - h_2^\star)(y)|\, dy \\
&\le \int_{-\infty}^{\infty}|(h_1^\star - h_2^\star)(y)|\, dy = \|h_1^\star - h_2^\star\|_1 \le \varepsilon.
\end{aligned}$$

Remark 6.4. The initial value problem of the higher-dimensional heat equation can be similarly formulated as:

$$\begin{cases} u_t = k\Delta u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = h(x), & x \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

and the energy functional can be similarly defined as

$$E_L(h,t) = \int_{B(0,L)} u_h(x,t)dx$$

where $B(0,L)$ denotes the ball in $\mathbb{R}^n$ centered at the origin with radius L and u_h denotes the solution of (6.1). A similar result as Theorem 6.1 holds and the proof is in principle the same as that of one-dimensional case by changing $\chi_{[-L,L]}$ to $\chi_{B(0,L)}$.

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