Symmetry and Free Boundary Points in a Class of Linear Ordinary Differential Equations

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Abstract

This note is concerned with the qualitative properties of the solutions of a class of linear ordinary differential equations. The existence and uniqueness of solutions are addressed, and properties of the graph of the solution when imposing some restrictions are derived. A new notion of derivative, called the force derivative, is introduced and an orthogonality result, between the force derivative of the solution and the force function, is obtained. All the important results are verified by numerical examples using MATLAB. Finally, an inequality result reminiscent of the famous G. Talenti's inequality is proved.

Key Words: Existence, uniqueness, symmetry, free boundary points, inequality
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1 Introduction

In physics, the catenary describes the curve assumed by a hanging flexible wire or chain when supported at its ends and acted upon by gravity. When the acting vertical force is generalized by $f(x)$, the vertical displacement denoted by u satisfies the equation:

$$\begin{cases} -u'' = f(x), & x \in (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{1}$$

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To further generalize the idea, consider there is horizontal force on the wire, which is denoted by $A(x)$. This note discusses some qualitative properties of the solutions of the following boundary value problem:

$$\begin{cases} -\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = f(x), & x \in (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{2}$$

Here, the function $A(x)$ and $f(x)$ are assumed to be smooth functions on the closed interval $[-1,1]$ and $A(x) \geq 1, f(x) \geq 0$; These assumptions guarantee the solution u will be classical; i.e. $u \in C^2(-1,1) \cap C^1[-1,1]$. More conditions shall be added to $A(x)$ and $f(x)$ in the following content in order to obtain desired results.

In what follows, this paper will first prove a basic result concerning the existence and uniqueness of solutions to (2). This is achieved by deriving an explicit formula for the solution u. Once this result is established, further conditions that both functions are even are imposed on $A(x)$ and $f(x)$. Such conditions shall be enforced throughout the rest of the note. As a consequence of the additional restrictions, the solution u will turn out to be even and non-increasing on $(0,1)$. This, in turn, gives rise to the concept of *free boundary points* of u. Let α be the value of $u(x)$. $\forall \alpha \in (0, u_M)$, these are the two endpoints ($u(x) = \alpha$) of the interval $\{x \in [-1,1] : u(x) \geq \alpha\}$; thanks to the even-ness of u, such points are well defined. The first main result is an implicit formula that computes the locations of the free boundary points. Using this implicit formula, one will be able to derive an interesting inequality reminiscent of the well known G. Talenti's inequality, see for example [2].

Another main result is concerned with particular perturbations of (2). More precisely, $f(x)$ in (2) shall be replaced by $f_t(x) = f(x-t)$, for small values of t, while keeping $A(x)$ intact. The corresponding family of solutions denoted $u_t(x)$ contribute to the introduction of a new notion of derivative that is called *force derivative* of u. The force derivative of u is denoted by $\dot{u}$ to differ from the usual derivative u'. It shall be proved that $\dot{u}$ is orthogonal to $f(x)$ in $L^2(-1,1)$; i.e.

$$\int_{-1}^{1} \dot{u}(x) f(x) dx = 0.$$

2 Basic results

In this section some basic results are collected about the solutions of the boundary value problem (2).

Theorem 2.1. *The boundary value problem (2) has a unique solution:*

$$u(x) = C\int_{-1}^{x} \frac{dy}{A(y)} - \int_{-1}^{x} \frac{F(y)}{A(y)} dy, \tag{3}$$

where $F(x) = \int_{-1}^{x} f(y)dy$ *and* C *is the constant given by the following formula*

$$C = \frac{\int_{-1}^{1} \frac{F(y)}{A(y)}dy}{\int_{-1}^{1} \frac{dy}{A(y)}} = \frac{\int_{-1}^{1}\int_{-1}^{y} \frac{f(s)}{A(y)}dsdy}{\int_{-1}^{1} \frac{dy}{A(y)}}.$$

Proof. Reduce the order of the differential equation in (2) from two to one by setting $v = u'$. This yields the following first order linear ODE

$$A(x)v' + a(x)v = -f(x), \tag{4}$$

where the notation $a(x) = A'(x)$ has been used. The reduced equation (4) can be readily solved, and its general solution, in conjunction with the boundary conditions $u(\pm 1) = 0$, will lead to the formula displayed in (3). □

Remark 2.1. This remark shows that the *uniqueness* of solutions to the boundary value problem (2) can be proved once the existence of solutions has been established. To this end, first show that if $f \equiv 0$ in (2), then u must be the zero function in $[-1, 1]$. So, begin it with

$$\begin{cases} -\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = 0, \quad x \in (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{5}$$

Multiplying the differential equation in (5) by u, integrating the result from -1 to 1, and applying integration by parts yields

$$\int_{-1}^{1} u(A(x)u')'dx = u(A(x)u')|_{-1}^{1} - \int_{-1}^{1} A(x)u'^2dx = 0 - \int_{-1}^{1} A(x)u'^2dx = 0, \tag{6}$$

where the boundary conditions $u(\pm 1) = 0$ are used. Since the integrand of the last integral in (6) is continuous, and non-negative, it can be deduced that $A(x)u'^2 = 0$ in $[-1, 1]$. Since $A(x) \geq 1$, it implies that $u' = 0$ throughout $[-1, 1]$. Whence, u must be constant. The boundary conditions $u(\pm 1) = 0$ ensure that $u = 0$ in $[-1, 1]$.

Next, assume (2) has two solutions, say, u_1 and u_2. It shall be proved that $u_1 = u_2$, and this completes the proof of uniqueness. Introduce the difference function $w(x) = u_1(x) - u_2(x)$, and observe that the following holds

$$\begin{cases} -\frac{d}{dx}\left(A(x)\frac{dw}{dx}\right) = 0, \quad x \in (-1,1) \\ w(\pm 1) = 0. \end{cases} \tag{7}$$

Using the argument in the beginning of this remark, it can be inferred that $w \equiv 0$. In other words, $u_1 = u_2$, which is the desired result.

Theorem 2.2. *If* f *is not identically zero in the interval* $[-1, 1]$*, then* u *is positive (i.e.* $u > 0$*) on* $(-1, 1)$*. Hence,* u *has a positive maximum denoted* u_M*.*

Proof. To derive a contradiction, first assume there exists $x^* \in (-1, 1)$ such that $u(x^*) = \min_{x\in[-1,1]} u(x) \leq 0$. Note that in this case $u'(x^*) = 0$. Next, integrate the differential equation in (2) from x^* to an arbitrary $x \in (x^*, 1)$ and obtain

$$-\int_{x^*}^{x} (A(t)u'(t))'dt = \int_{x^*}^{x} f(t)dt. \tag{8}$$

Since f is non-negative, the last equation implies that $u'(x)$ is non-positive for every $x \in (x^*, 1)$. By using a similar argument, one can verify that $u'(x)$ is non-negative for every $x \in (-1, x^*)$. Whence, u must be constant throughout $[-1, 1]$. This is a contradiction due to the boundary conditions $u(\pm 1) = 0$ and the fact that u cannot be the zero function. □

As noted earlier, the conditions imposed on A and f gradually become stricter. In the next lemma it shall be assumed that both of these functions are even.

Theorem 2.3. *If f and A are even, then u is even and non-increasing on* $(0, 1)$.

Proof. Let $w(x) = u(x) - u(-x)$. Then observe that $w(\pm 1) = 0$. Differentiate $w(x)$ and get

$$w'(x) = u'(x) + u'(-x). \tag{9}$$

Multiply (9) by $A(x)$, differentiate the result, and use the differential equation in (2) to yield

$$\begin{aligned}(A(x)w'(x))' &= (A(x)u'(x) + A(-x)u'(-x))' \\ &= f(x) - f(-x) = 0,\end{aligned}$$

because A and f are even functions. Using the uniqueness result proved in Remark 2.1, one can infer $w = 0$. This, in turn, implies that $u(x) = u(-x)$, as desired.

To prove the second part of the assertion, fix $x_0 \in (0, 1)$, and integrate the differential equation in (2) from 0 to x_0:

$$\int_0^{x_0} -(A(x)u')'dx = \int_0^{x_0} f(x)dx. \tag{10}$$

Note that the integral on the right hand side of (10) is non-negative, and the integral on the left hand side can be explicitly evaluated to get

$$-(A(x_0)u'(x_0) - A(0)u'(0)) \geq 0.$$

Since u is even, u' is odd and $u'(0) = 0$. This, coupled with the last inequality, implies

$$u'(x_0) \leq 0.$$

Since x_0 is arbitrary, it can be deduced that u is non-increasing on $(0, 1)$. This completes the proof of the lemma. Particularly, if $f(x) > 0$ for $x \in (0, 1)$, then u is strictly decreasing on $(0, 1)$.

□

The rest of content is based on the following assumptions:
1. f and A are even.
2. $f \geq 0$, $A \geq 1$.
3. There exists some $x \in [-1, 1]$, such that $f(x) > 0$, i.e. f is not constantly 0. **(A)**

Theorem 2.4. *Let $\hat{x} = inf\{x > 0 : f(x) > 0\}$, then $u = u_M$ on $[-\hat{x}, \hat{x}]$. On the other hand, u is strictly decreasing on $[\hat{x}, 1]$. Hence, $u : (\hat{x}, 1) \to (0, u_M)$ defines a homeomorphism. Note that $u_M = \max_{x\in[-1,1]} u(x)$ here.*

Proof. Fix $x \in (0, 1)$. Integrating the differential equation in (2) from 0 to x gives

$$-\int_0^x (A(t)u'(t))'dt = \begin{cases} \int_0^x f(t)dt = 0 & x \in (0, \hat{x}) \\ \\ \int_{\hat{x}}^x f(t)dt > 0 & x \in (\hat{x}, 1). \end{cases} \tag{11}$$

Recalling $u'(0) = 0$, the integral on the left hand side of the inequality (11) is equal to $-A(x)u'(x)$. Hence, one can obtain the desired results. □

Theorem 2.4 infers the restriction of u to the interval $(\hat{x}, 1)$ is invertible, i.e. $(u|_{(\hat{x},1)})^{-1}$ exists. Henceforth, this paper abuses the notation and writes u^{-1} instead of $(u|_{(\hat{x},1)})^{-1}$.

3 Main results

The main results of this note are gathered in this section that splits into three subsections.

3.1 Free boundary points and their locations

Definition 3.1. Let u be the solution to the boundary value problem (2). For $\alpha \in (0, u_M)$, the α-cut of u, denoted by u_α, is define as follows.

$$u_\alpha = \{x \in (-1, 1) : u(x) \geq \alpha\}.$$

The boundary points of u_α, denoted by ∂u_α, are called the *free boundary points* of u:

$$\partial u_\alpha = \{\pm u^{-1}(\alpha)\}.$$

Theorem 3.1. *If u is the solution to the boundary value problem (2), then for $\alpha \in (0, u_M)$, the free boundary point $u^{-1}(\alpha)$ is implicitly computed by*

$$\alpha = \int_{u^{-1}(\alpha)}^{1} \frac{F(y)}{A(y)} dy, \tag{12}$$

where $F(y) = \int_0^y f(x)dx$.

Proof. First integrate the differential equation in (2) over $u_\alpha = [-u^{-1}(\alpha), u^{-1}(\alpha)]$ which yields

$$-\int_{-u^{-1}(\alpha)}^{u^{-1}(\alpha)} (A(x)u'(x))'dx = \int_{-u^{-1}(\alpha)}^{u^{-1}(\alpha)} f(x)dx. \tag{13}$$

Recalling that u' is odd, (13) leads to

$$-A(u^{-1}(\alpha))u'(u^{-1}(\alpha)) = \int_0^{u^{-1}(\alpha)} f(x)dx. \tag{14}$$

By the definition of the derivative of inverse functions, (14) yields

$$-\frac{A(u^{-1}(\alpha))}{(u^{-1})'(\alpha)} = \int_0^{u^{-1}(\alpha)} f(x)dx.$$

Hence,

$$-(u^{-1})'(\alpha) = \frac{A(u^{-1}(\alpha))}{\int_0^{u^{-1}(\alpha)} f(x)dx}.$$

Next, use the change of variable $y = u^{-1}(\alpha)$ to obtain

$$-y'(\alpha) = \frac{A(y)}{\int_0^y f(x)dx} = \frac{A(y)}{F(y)} = -\frac{dy}{d\alpha}, \tag{15}$$

where $F(y) = \int_0^y f(x)dx$. Finally, by integrating (15), one can get

$$\int_{u^{-1}(\alpha)}^{1} \frac{F(y)}{A(y)} dy = -\int_1^{u^{-1}(\alpha)} \frac{F(y)}{A(y)} dy = \int_0^\alpha ds = \alpha.$$

□

Example 1. Here is an example that confirms the validity of the formula (12). Consider the following BVP:

$$\begin{cases} -\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = f(x), & x \in (-1, 1) \\ u(\pm 1) = 0. \end{cases} \tag{16}$$

$A(x) = cos(x^2) + 1,\ f(x) = cosx$. One can obtain a numerical solution as Figure 1(a):

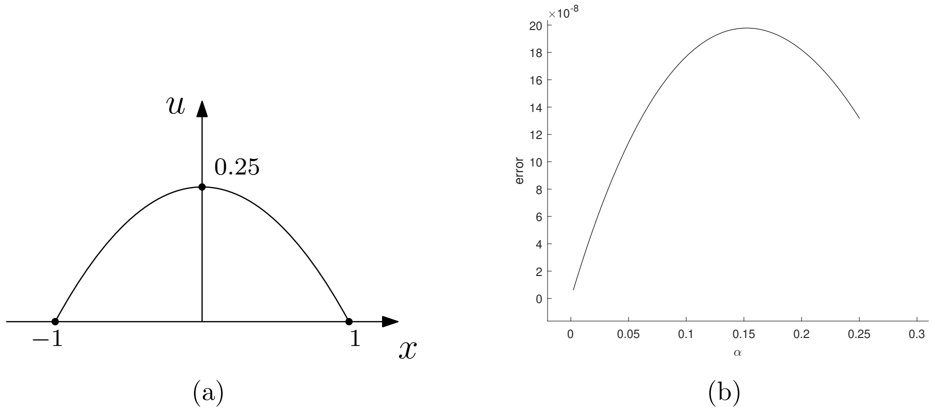

Figure 1: (a)Numerical result of u in example 1; (b)Graph of e versus α

Let e be the error of true α and numerical α calculated by formula 12, i.e.

$$e = \alpha - \int_{u^{-1}(\alpha)}^{1} \frac{F(y)}{A(y)} dy.$$

The graph Figure 1(b) plots the error with different values of α shows that the error is sufficiently small. Thus,

$$\int_{u^{-1}(\alpha)}^{1} \frac{F(y)}{A(y)} dy = \alpha.$$

Example 2. Consider the following boundary value problem

$$\begin{cases} -((1+x^2)u')' = 1 & x \in (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{17}$$

Here, $A(x) = 1 + x^2$ and $f(x) = 1$, hence $F(y) = \int_0^y f(x)dx = y$. From the formula (12) one can get

$$\alpha = \int_{u^{-1}(\alpha)}^{1} \frac{y}{1+y^2} dy.$$

This, in turn, gives $u^{-1}(\alpha) = \sqrt{2e^{-2\alpha} - 1}$. Noted that since $2e^{-2\alpha} - 1$ is non-negative, it can be inferred $\alpha \leq \frac{\ln 2}{2}$. So, in particular, $u_M = \frac{\ln 2}{2}$.

The next definition concerns the force function $f(x)$ in (2).

Definition 3.2. The function f is said to be of *Positon type* if $f(0) > 0$.

Corollary 3.2. *If f is of Positon type, then by Theorem 3.1:*

$$u_M = \alpha = \int_0^1 \frac{F(y)}{A(y)} dy.$$

3.2 The force derivative and orthogonality

Throughout this section, such assumption is made that, for a small positive ϵ, the force function f satisfies

$$\{x \in [-1,1] : f(x) > 0\} \subseteq [-1+\epsilon, 1-\epsilon]. \tag{\textbf{B}}$$

Next, consider the following perturbations of f:

$$f_t(x) = f(x-t), \qquad 0 < t < \epsilon.$$

This models the time-dependent acting force $f(x)$ along the wire. In a sufficiently small length of time, $f(x)$ is perturbed towards the right. In reality the slight translation of force might happen. Note that under the assumption, $\{x \in [-1,1] : f_t(x) > 0\} \subseteq [-1,1]$. Let u_t denote the solution of (2) with f replaced by f_t; i.e. u_t is the solution of the following:

$$\begin{cases} -(A(x)u_t')' = f_t(x), & \text{in } (-1,1) \\ u_t(\pm 1) = 0 \end{cases}$$

In the next definition, a new way shall be defined to differentiate u. In order not to confuse this type of derivative with the usual one, this paper uses "dot" to indicate this particular derivative.

Definition 3.3. Let u be the solution of (2) while u_t be the solutions of (2) with f replaced by f_t. The *force derivative* of u is denoted by $\dot{u}$ and defined as

$$\dot{u}(x) = \lim_{t\to 0} \frac{u_t(x) - u(x)}{t}. \tag{18}$$

An immediate question rises that whether or not the limit in (18) exists. This question is addressed in the next lemma.

Lemma 3.3. *The limit in (18) exists and its value is finite.*

Proof. From Theorem 2.1 one can get,

$$u_t(x) = C(t)\int_{-1}^{x} \frac{dy}{A(y)} - \int_{-1}^{x} \frac{F_t(y)}{A(y)}dy,$$

where

$$F_t(x) = \int_{-1}^{x} f_t(y)dy = \int_{-1}^{x} f(y-t)dy \qquad \text{and} \qquad C(t) = \frac{\int_{-1}^{1}\int_{-1}^{y} \frac{f_t(s)}{A(y)}dsdy}{\int_{-1}^{1} \frac{dy}{A(y)}}.$$

Here

$$\frac{u_t(x) - u(x)}{t} = H(x) \int_{-1}^{1} G(y)dy - \int_{-1}^{x} G(y)dy,$$

$$H(x) = \frac{\int_{-1}^{x} \frac{dy}{A(y)}}{\int_{-1}^{1} \frac{dy}{A(y)}} \qquad \text{and} \qquad G(y) = \int_{-1}^{y} \frac{1}{A(y)} \frac{f(s-t) - f(s)}{t} ds.$$

Note that $f(x)$ are smooth, so one can derive

$$\dot{u}(x) = -\frac{\int_{-1}^{x} \frac{dy}{A(y)}}{\int_{-1}^{1} \frac{dy}{A(y)}} \int_{-1}^{1} \int_{-1}^{y} \frac{f'(s)}{A(y)} dsdy + \int_{-1}^{x} \int_{-1}^{y} \frac{f'(s)}{A(y)} dsdy. \tag{19}$$

□

Theorem 3.4. *Let u be the solution of (2), and $\dot{u}$ be its force derivative. Then $\dot{u}$ is L^2-orthogonal to the force function $f(x)$; i.e.*

$$\langle \dot{u}, f \rangle_{L^2(-1,1)} := \int_{-1}^{1} \dot{u}(x) f(x)\, dx = 0. \tag{20}$$

Proof. Begin the proof by recalling the boundary value problem

$$\begin{cases} -(A(x)u_t'(x))' = f_t(x) & \text{in } (-1,1) \\ u(\pm 1) = 0. \end{cases} \tag{21}$$

Consider a test function $\varphi \in C_0^\infty(-1,1)$. Multiply the differential equation in (21) by φ and integrate the result over the interval $(-1,1)$ to obtain

$$-\int_{-1}^{1} \varphi(A(x)u_t')'dx = \int_{-1}^{1} f_t(x)\varphi(x)dx. \tag{22}$$

Applying integration by parts twice to the integral on the left hand side of (22), and keeping in mind that $\varphi(\pm 1) = 0$, one can get

$$-\int_{-1}^{1} u_t(x)(A(x)\varphi'(x))'dx = \int_{-1}^{1} f_t(x)\varphi(x)dx. \tag{23}$$

Similarly, one can derive

$$-\int_{-1}^{1} u(x)(A(x)\varphi'(x))'dx = \int_{-1}^{1} f(x)\varphi(x)dx. \tag{24}$$

From (23) and (24) it can be found that

$$\lim_{t\to 0} \int_{-1}^{1} \frac{u_t(x) - u(x)}{t}(A(x)\varphi'(x))'dx = -\lim_{t\to 0} \int_{-1}^{1} \frac{f_t(x) - f(x)}{t}\varphi(x)dx. \tag{25}$$

Changing the order of limits and integrals in (25) gives

$$\int_{-1}^{1} \dot{u}(x)(A(x)\varphi'(x))'dx = \int_{-1}^{1} f'(x)\varphi(x)dx. \tag{26}$$

Apply integration by parts twice to the left hand side integral of (26). Note that $\dot{u}$ is twice continuously differentiable, and that $\varphi(\pm 1) = 0$, then one can obtain

$$\int_{-1}^{1} (A(x)\dot{u}'(x))'\varphi(x)dx = \int_{-1}^{1} f'(x)\varphi(x)dx, \quad \forall \varphi \in C_0^{\infty}(-1, 1). \tag{27}$$

From (27) and the trivial facts that $\dot{u}(\pm 1) = 0$, one can derive

$$\begin{cases} (A(x)\dot{u}'(x))' = f'(x), & x \in (-1, 1) \\ \dot{u}(\pm 1) = 0. \end{cases} \tag{28}$$

Next, multiply the differential equation in (28) by u and integrate the result on $(-1, 1)$. Because $u(\pm 1) = 0$, the following identity holds:

$$-\int_{-1}^{1} A(x)\dot{u}'(x)u'(x)dx = \int_{-1}^{1} f'(x)u(x)dx. \tag{29}$$

Since $u(x)$ is even, and $f'(x)$ odd, the integral $\int_{-1}^{1} f'(x)u(x)dx$ vanishes. (29) implies that

$$\int_{-1}^{1} A(x)\dot{u}'(x)u'(x)dx = 0. \tag{30}$$

Multiply the differential equation in (2) by $\dot{u}$, integrate the result on $(-1, 1)$, and apply integration by parts, to get

$$\int_{-1}^{1} A(x)\dot{u}'(x)u'(x)dx = \int_{-1}^{1} f(x)\dot{u}(x)dx. \tag{31}$$

The assertion (20) follows from (30) and (31). □

Example 3. This example follows from Example 1, with $A(x) = cos(x^2) + 1$, $f(x) = cosx$. It demonstrates the existence of $\dot{u}$ as well as its orthogonality to f.

Approximate $\dot{u}$ with a converging sequence $\{\dot{u}_{t_i}\}$, where $t_i = 10^{-i}$, and $\dot{u}_{t_i}$ is defined as follows:

$$\dot{u}_{t_i}(x) = \frac{u_{t_i}(x) - u(x)}{t_i} \tag{32}$$

If $\dot{u}$ exists, $\dot{u}_{t_i}$ converges to $\dot{u}$ as i increases. From Figure 2, it is clear that $\dot{u}_{t_i}$ converges. Hence the existence of $\dot{u}$i s verified, which i s consistent with Lemma 3.3. Use this $\dot{u}_{t_6}$as a numerical approximation of $\dot{u}$ and look into the orthogonality property in the Hilbert space $L^2(-1,1)$. The inner product of $\dot{u}_{t_6}$ and f is found to be of 10^{-7} magnitude, which is consistent with Theorem 3.4.

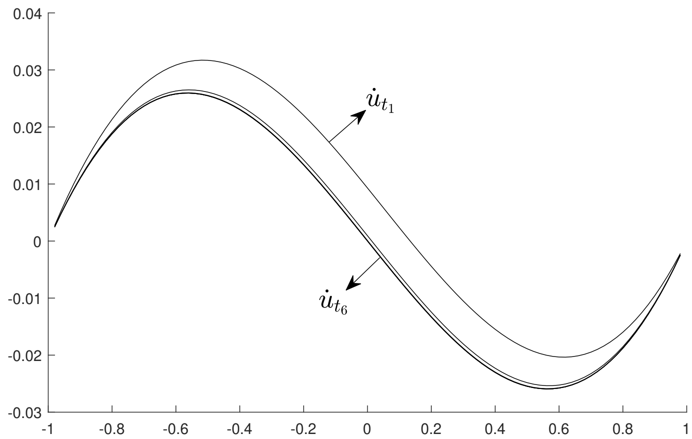

Figure 2: Behavior of $\dot{u}_{t_i}$. Here when $t = t_6 = 10^{-6}$, the inner product of $\dot{u}_{t_6}$ and f is 1.5159×10^{-7}.

3.3 Schwarz symmetrization and estimates

Throughout this section the assumption of the even-ness of function hold for $A(x)$ and $f(x)$, except that the assumption **(B)** is no longer enforced. Recall the definition of the Schwarz symmetrization of a function $h : [-1,1] \to [0,1]$.

Definition 3.4. For a smooth function $h : [-1,1] \to [0,\infty)$, the *Schwarz symmetrization* of h is denoted by $h^\sharp$ such that $h^\sharp : [-1,1] \to [0,\infty)$ uniquely satisfies the following conditions:

i) The length of $h^\sharp_\alpha$, the α-cut of $h^\sharp$, is equal to the length of h_α;
ii) The function $h^\sharp$ is even, and non-increasing on the interval $[0,1]$.

A useful property of Schwarz symmetrization, referring to [1], is the following inequality:

$$\int_{-\gamma}^{\gamma} h(t)dt \le \int_{-\gamma}^{\gamma} h^\sharp(t)dt, \qquad \forall \gamma \in (0,1). \tag{33}$$

It is now ready to state and prove the last main result of this note which should be reminiscent of the famous G. Talenti's inequality, in reference to [2].

Theorem 3.5. *Let u be the unique solution of the boundary value problem (2). Let $v(x)$ be the solution of the following boundary value problem:*

$$\begin{cases} -Z''(x) = f^\sharp(x) \quad in\ (-1,1) \\ Z(\pm 1) = 0, \end{cases} \tag{34}$$

where $f^\sharp(x)$ denotes the Schwarz symmetrization of $f(x)$. Then

$$u(x) \leq v(x) \qquad \forall\, x \in (-1,1). \tag{35}$$

Proof. Set $v_M = \max_{x\in[-1,1]} v(x)$, and as usual u_M denotes the maximum value of u on $[-1,1]$. Fix $\alpha \in (0, u_M \wedge v_M)$, where "$a \wedge b$" denotes the minimum of a and b. Then, by (12),

$$\int_{u^{-1}(\alpha)}^{1} \frac{F(y)}{A(y)} dy = \int_{v^{-1}(\alpha)}^{1} \tilde{F}(y) dy, \tag{36}$$

where

$$F(y) = \int_0^y f(x)dx \qquad \text{and} \qquad \tilde{F}(y) = \int_0^y f^\sharp(x)dx.$$

Inequality (33) infers $F(y) \leq \tilde{F}(y)$. Together with $A(x) \geq 1$, it can be deduced from (36) that

$$\int_{v^{-1}(\alpha)}^{1} \tilde{F}(y)dy \leq \int_{u^{-1}(\alpha)}^{1} \tilde{F}(y)dy,$$

hence $u^{-1}(\alpha) \leq v^{-1}(\alpha)$. This, in turn, implies that

$$[-u^{-1}(\alpha), u^{-1}(\alpha)] \subseteq [-v^{-1}(\alpha), v^{-1}(\alpha)], \quad \forall \alpha \in (0, u_M \wedge v_M). \tag{37}$$

Since $f(x)$ is positive at some point, $f^\sharp(0)$ is a positon type. Hence $v(x)$ is strictly decreasing on $[0,1]$. Thus, $v^{-1}(v_M) = 0$. If $u_M \geq v_M$, then $v_M \in (0, u_M \wedge v_M]$. $u^{-1}(v_M) \leq 0$, so $u_M = u(0) = v_M$. Together with (37), these complete the proof of (35). □

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