L^1 Regularization for Compact Support

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Abstract

In this paper, we solve a conjecture of Osher and Yin that L^1 regularization of certain minimization problems ensures compact support of minimizers. In particular, we consider a class of minimization problems and show that adding an L^1 term forces compact support of solutions given certain sufficient conditions namely that the given forcing function sufficiently decays toward $-\infty$ and ∞. It turns out that the condition found is sharp.

1 Introduction

Borrowing heavily from Osher and Yin [8] (see also [2]), we consider the context of signals. Signal processing has become increasingly important in many fields. As vectors in $\mathbb{R}^n$, these can represent any number of types of data like sound, wireless communication, images, video, and 4D CT.

Of particular importance are "sparse" signals, vectors which have few nonzero elements or have few nonzero elements after a specific basis change. Sparsity is a key component in signal processing methods like compressed sensing, regularized regression, and regularized inverse problems.

As such, it is instructive to look at the following minimization problem. Let A be a $m \times n$ matrix and $b \in \mathbb{R}^m$ a vector. Suppose we wish to find the sparsest (most zero terms) solution to the equation $Ax = b$. We can denote this mathematically as the problem of minimizing the l_0 norm (the number of zero terms) of x subject to $Ax = b$. This is in general fairly difficult. An easier problem is the problem of minimizing the l^1 norm (the sum of the magnitude of terms). It turns out that there are relatively simple conditions for which this alternate problem "recovers" the true solution. In other words, it is understood exactly when this method finds the sparsest solution. This is good news because it allows for relatively rapid solutions to the l_0 minimization problem. This works precisely because the l^1 norm "forces" sparsity of the vector.

If we transition to the infinite dimensional case of functions, the l^1 norm is analogous to the L^1 norm and sparsity is analogous to compact support. Thus, it is hoped that including an L^1 norm "forces" compact support in minimization problems [8, p. 3].

In what follows, $H^1(\mathbb{R})$ is the space of all absolutely continuous functions $u : \mathbb{R} \to \mathbb{R}$ such that $u \in L^2(\mathbb{R})$ and $u' \in L^2(\mathbb{R})$, endowed with the norm

$$||u||_{H^1} := ||u||_{L^2} + ||u'||_{L^2}.$$

Note that the derivative of an absolutely continuous function exists $\mathcal{L}^1$-a.e. in $\mathbb{R}$. We refer to [7] for more details on the space $H^1(\mathbb{R})$. Here $\mathcal{L}^1$ is the Lebesgue measure in $\mathbb{R}$.

1.1 The First Problem

We begin by considering the following problem. For $f \in L^2(\mathbb{R})$ and $u \in H^1(\mathbb{R})$, define the following functional

$$F(u) := \int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 - f(x)u(x)dx.$$

As motivation, u may represent some configuration such as electron density or chemical concentration. Then, F can be interpreted as the energy associated with each configuration. As such, we want to find the minimum across all possible u giving the following problem.

Problem 1. *Find*

$$\min_{u\in H^1(\mathbb{R})} F(u) = \min_{u\in H^1(\mathbb{R})} \int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 - f(x)u(x)dx.$$

We note that the functional inside the integral is C^2, so the Euler-Lagrange equation from Calculus of Variations [3] becomes

$$u''(x) = -f(x)$$

for any minimizer u of F. This is not a very satisfying outcome. Supposing f nonzero on some interval I, then u'' is nonzero on that interval and thus u cannot be identically zero on that interval. This means that in many cases, u does not have compact support. The question becomes: how can we alter the problem to get compact support?

1.2 A Better Problem

To this end, we consider the following:

Definition 2. *Define*

$$V = \{u : \mathbb{R} \to \mathbb{R} \text{ absolutely continuous and } ||u||_{L^1} + ||u'||_{L^2} < \infty\}$$

a normed space with the natural norm given by

$$||u||_V = ||u||_{L^1} + ||u'||_{L^2} = \int_{\mathbb{R}} |u(x)|dx + \left(\int_{\mathbb{R}} (u'(x))^2 dx\right)^{\frac{1}{2}}$$

and

$$F(u) := \int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 - f(x)u(x) + \frac{1}{\mu}|u(x)|dx$$

with $f \in L^2(\mathbb{R})$, $\mu > 0$, and $u \in V$. Note the similarity to and differences with the problem above especially the L^1 term. Note also that the integral in the definition of F can be changed to be over the support of u giving a free boundary problem. See, e.g., the book of Kinderlehrer and Stampacchia for a comprehensive treatment of free boundary problems. [6].

Problem 3. *Find*

$$\min_{u \in V} F(u) = \min_{u \in V} \int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 - f(x)u(x) + \frac{1}{\mu}|u(x)|dx.$$

In this paper, we only consider the problem in one dimension. This problem is discussed by Osher and Yin in one dimension [8] and by Caflish, Osher, Schaeffer, and Tran in higher dimensions [2]. Figures 1 and 2 in that paper shows the computational solution to one such problem in two dimensions.

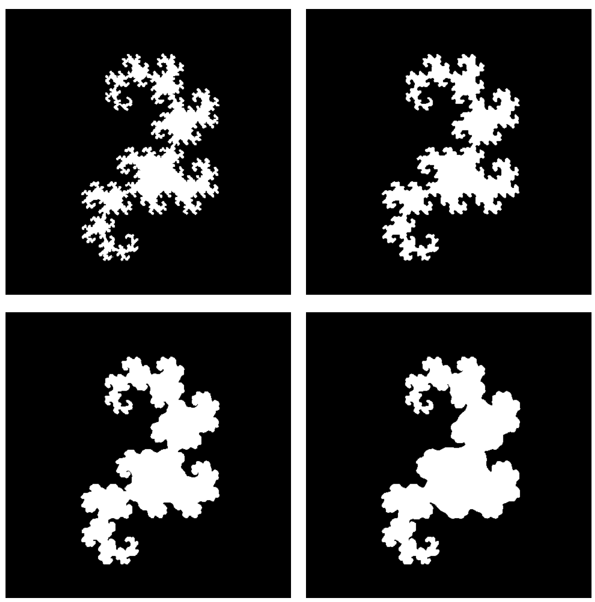

Figure 1: An iterated numerical solution to this problem in two dimensions with $f = 1.5\chi_S$ where S is shown top left and $\frac{1}{\mu} = 1$. The support of minimizer u is shown bottom right with the support of intermediate steps left to right, top to bottom. Courtesy of Caflish, Osher, Schaeffer, and Tran [2]

Now that we have the problem defined, we can present the main results found. First, we have existence of a minimizer

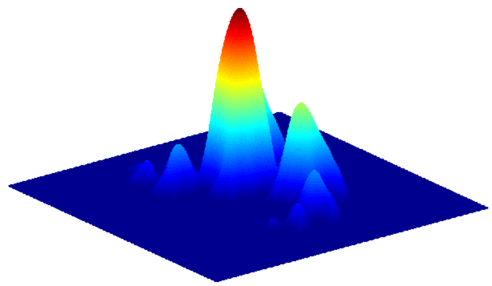

Figure 2: The graph of numerical solution u in Figure 1. Courtesy of Caflish, Osher, Schaeffer, and Tran [2]

Theorem 4 (Existence)**.** *Let $f \in L^2(\mathbb{R})$ and $\mu > 0$. There is $u_0 \in V$ so that*

$$F(u_0) \leq F(u)$$

for all $u \in V$. Furthermore, this minimizer u is unique.

In what follows, we take f to be a function and not an equivalence class of functions. Thus, it makes sense to talk about limits. Knowing that minimizers of the problem exist, we can present sufficient conditions for compact support of minimizers. That is,

Theorem 5 (Compact support)**.** ($\star$) *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. If*

$$\limsup_{x \to \infty} |f(x)| < \frac{1}{\mu} \qquad \text{and} \qquad \limsup_{x \to -\infty} |f(x)| < \frac{1}{\mu}, \tag{1}$$

then any minimizer u of F has compact support on the real line.

We call these conditions (1), the limsup conditions. A corollary follows easily

Corollary 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. Assume $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$ and take any $\mu > 0$. Then, any minimizer u of F has compact support on the real line.*

Furthermore, we can bound the measure of the support on the real line

Theorem 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. Let $u \in V$ be a minimizer of F with compact support. Then, the measure of the support of u is bounded by $\mu^2 ||f||_{L^2}^2$.*

Lastly we provide a bound on the support based only on f, a given value.

Theorem 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. Let $\varepsilon > 0$ and suppose that there are x_0, x_1 so that*

$$\sup_{x \geq x_0} |f(x)| \leq \frac{1}{\mu} - \varepsilon \qquad \text{and} \qquad \sup_{x \leq x_1} |f(x)| \leq \frac{1}{\mu} - \varepsilon.$$

Then for any minimizer u of F, the support of u is bounded between $x_1 - \mu^2(||f||_{L^2})^2$ on the left and $x_0 + \mu^2(||f||_{L^2})^2$ on the right.

Taking these results together, we have proven the conjecture by Osher and Yin [8] that the L^1 regularization provides compact support. In the higher dimensional case, we refer to the paper of Brezis [1] in which he proved that solutions of variational inequalities have compact support using the theory of maximal monotone operators.

2 Existence of a Minimizer

In this section, we prove Theorem 4. To prove this, we need a few preliminaries. First, we present Theorem 7.34 in Leoni [7] without proof here

Lemma 9 (Interpolation Inequality)**.** *For all $u \in V$,*

$$||u||_{L^2} \leq \frac{1}{l^{\frac{1}{2}}}||u||_{L^1} + l||u'||_{L^2}$$

for all $0 < l < \infty$.

Using this lemma, we can prove the following useful proposition

Proposition 10. *Let $f \in L^2(\mathbb{R})$ and $\mu > 0$. Then,*

$$\inf_{u \in V} F(u) > -\infty$$

Proof. To show this, fix $\mu > 0$ and $f \in L^2(\mathbb{R})$ and do the following computation using first Hölder's inequality and then Lemma 9 above

$$\begin{aligned}
F(u) &= \int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 - f(x)u(x) + \frac{1}{\mu}|u(x)|dx \\
&\geq \frac{1}{2}||u'||_{L^2}^2 - ||f||_{L^2}||u||_{L^2} + \frac{1}{\mu}||u||_{L^1} \\
&\geq \frac{1}{2}||u'||_{L^2}^2 - ||f||_{L^2}\left(\frac{1}{l^{\frac{1}{2}}}||u||_{L^1} + l||u'||_{L^2}\right) + \frac{1}{\mu}||u||_{L^1}
\end{aligned}$$

for all $0 < l < \infty$. In the case that $||f||_{L^2} = 0$, then we get

$$F(u) \geq \frac{1}{2}||u'||_{L^2}^2 + \frac{1}{\mu}||u||_{L^1} \geq \frac{1}{2}||u'||_{L^2}^2 \geq 0$$

and there is nothing left to prove. Otherwise, we take $l = \mu^2||f||_{L^2}^2$. This gives

$$\begin{aligned}
F(u) &\geq \frac{1}{2}||u'||_{L^2}^2 - ||f||_{L^2}\left(\frac{1}{\mu||f||_{L^2}}||u||_{L^1} + \mu^2||f||_{L^2}^2||u'||_{L^2}\right) + \frac{1}{\mu}||u||_{L^1} \\
&= \frac{1}{2}||u'||_{L^2}^2 - \frac{1}{\mu}||u||_{L^1} - \mu^2||f||_{L^2}^3||u'||_{L^2} + \frac{1}{\mu}||u||_{L^1} \qquad (2) \\
&= \frac{1}{2}||u'||_{L^2}^2 - \mu^2||f||_{L^2}^3||u'||_{L^2}.
\end{aligned}$$

Hence, F is bounded from below by a convex quadratic in $||u'||_{L^2}$. This attains a minimum and so,

$$\inf_{u \in V} F(u) \geq \inf_{u \in V} \frac{1}{2}||u'||_{L^2}^2 - \mu^2||f||_{L^2}^3||u'||_{L^2} > -\infty.$$

■

This is important because it shows that F may indeed have a minimizer u. Further, for any $u \in V$ we have that

$$\int_{\mathbb{R}} \frac{1}{2}(u'(x))^2 dx = \frac{1}{2}||u'||_{L^2}^2 < \infty$$

and

$$\int_{\mathbb{R}} -f(x)u(x)dx \leq \int_{\mathbb{R}} |f(x)u(x)|dx \leq ||f||_{L^2}||u||_{L^2} < \infty$$

using Hölder's inequality and

$$\int_{\mathbb{R}} \frac{1}{\mu}|u(x)|dx = \frac{1}{\mu}||u||_{L^1} < \infty$$

which gives $F(u) < \infty$ for all $u \in V$. Note that the functional is not C^1 (absolute value has no derivative at 0) which prevents using the Euler-Lagrange equations. However, a powerful property that we can use is convexity.

Proposition 11. *Let $f \in L^2(\mathbb{R})$ and $\mu > 0$. Then, F is strictly convex.*

Proof. For $u_1, u_2 \in V$ and $\theta \in [0, 1]$, compute

$$\begin{aligned}
F(\theta u_1 + (1-\theta)u_2) &= \int_{\mathbb{R}} \frac{1}{2}(\theta u_1' + (1-\theta)u_2')^2 - \theta f u_1 - (1-\theta) f u_2 + \frac{1}{\mu}|\theta u_1 + (1-\theta)u_2| dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2}(\theta(u_1')^2 + (1-\theta)(u_2')^2) - \theta f u_1 - (1-\theta) f u_2 \\
&\qquad + \frac{1}{\mu}(|\theta u_1| + |(1-\theta)u_2|)dx \\
&= \left(\int_{\mathbb{R}} \frac{1}{2}\theta(u_1')^2 - \theta f u_1 + \frac{1}{\mu}|\theta u_1| dx\right) \\
&\qquad + \left(\int_{\mathbb{R}} \frac{1}{2}(1-\theta)(u_2')^2 - (1-\theta) f u_2 + \frac{1}{\mu}|(1-\theta)u_2| dx\right) \\
&= \theta F(u_1) + (1-\theta)F(u_2)
\end{aligned}$$

using the convexity of $t \mapsto t^2$ and the convexity of absolute value in the inequality step. Since $u \mapsto \int_{\mathbb{R}} u^2 dx$ is strictly convex, F is strictly convex. ■

The following lemma ensures that if F is bounded on a sequence, the sequence is bounded in V.

Lemma 12. *If $\{u_n\}_n$ is a sequence in V so that $\{F(u_n)\}_n$ is bounded from above, then $\{||u_n||_V\}_n$ is bounded.*

Proof. Compute for $u \in V$,

$$\begin{aligned}
F(u) &= \int \frac{1}{2}(u')^2 - fu + \frac{1}{\mu}|u| dx \\
&= \frac{1}{2\mu}\left(||u||_V - ||u||_{L^1} - ||u'||_{L^2}\right) + \int \frac{1}{2}(u')^2 - fu + \frac{1}{\mu}|u| dx \\
&= \frac{1}{2\mu}||u||_V + \left(\int \frac{1}{2}(u')^2 - fu + \frac{1}{2\mu}|u| dx\right) - \frac{1}{2\mu}||u'||_{L^2} \\
&\geq \frac{1}{2\mu}||u||_V + \frac{1}{2}||u'||_{L^2}^2 - (2\mu)^2||f||_{L^2}^3||u'||_{L^2} - \frac{1}{2\mu}||u'||_{L^2}
\end{aligned}$$

using inequality (2) with 2μ. The quadratic in $||u'||_{L^2}$ attains a minimum, so

$$\begin{aligned} F(u) &\geq \frac{1}{2\mu}||u||_V + \frac{1}{2}||u'||_{L^2}^2 - (2\mu)^2||f||_{L^2}^3||u'||_{L^2} - \frac{1}{2\mu}||u'||_{L^2} \\ &\geq \frac{1}{2\mu}||u||_V + C \end{aligned}$$

Because $\{F(u_n)\}_n$ is bounded from above, $\{\frac{1}{2\mu}||u_n||_V + C\}_n$ is also bounded from above which implies that $\{||u_n||_V\}_n$ is bounded from above. ■

Proposition 13. *If $u \in V$, then $u \in L^2(\mathbb{R})$ and $u' \in L^2(\mathbb{R})$.*

Proof. In the interpolation inequality (Lemma 9), taking $l = 1$ we find

$$||u||_{L^2} \leq ||u||_{L^1} + ||u'||_{L^2} = ||u||_V < \infty \tag{3}$$

so that we have $u \in L^2(\mathbb{R})$. Furthermore, $||u'||_{L^2} \leq ||u||_V < \infty$ giving $u' \in L^2(\mathbb{R})$. So, $u \in H^1(\mathbb{R})$. ■

The following propositions are presented without proof. See [7].

Proposition 14. *$L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ are Hilbert spaces.*

Proposition 15. *Let Y be a Hilbert space. Let $\{u_n\}_n$ be a bounded sequence in Y. Then, there exist $u \in Y$ and a subsequence $\{u_{n_k}\}_k$ so that $\{u_{n_k}\}_k$ weakly converges to u.*

We say that $\{u_n\}_n$ weakly converges to u or $u_n \rightharpoonup u$ if for every $v \in Y$, $\langle u_n, v\rangle \to \langle u, v\rangle$ as $n \to \infty$. If we let $Y = L^2(\mathbb{R})$, for every $g \in L^2(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} u_n g dx = \int_{\mathbb{R}} u g dx.$$

Proposition 16. *Let Y be a Hilbert space and let $\hat{F} : Y \to [-\infty, \infty]$ be a convex lower semicontinuous function. If $y_k \rightharpoonup y \in Y$, then*

$$\liminf_{k\to\infty} \hat{F}(y_k) \geq \hat{F}(y).$$

We are now ready to prove existence.

Proof of Theorem 4. We know that $\inf_{u\in V} F(u) \in \mathbb{R}$. Thus, we can find a sequence $\{u_n\}_n$ in V so that

$$\lim_{n\to\infty} F(u_n) = \inf_V F.$$

As $F(u_n)$ attains a limit in $\mathbb{R}$, the sequence $\{F(u_n)\}_n$ is bounded from above. By Lemma 12, we know that $\{||u_n||_V\}_n$ is bounded. Looking at inequality (3), we get that $\{||u_n||_{L^2}\}_n$ is bounded. Furthermore, $||u_n'||_{L^2} \leq ||u_n||_V$ is bounded as well. Applying Proposition 15 in $H^1(\mathbb{R})$ allows us to find a subsequence $\{u_{n_k}\}_k$ that weakly converges to some $u \in L^2$. We note that $\{||u_{n_k}'||_{L^2}\}_k$ is bounded, so we can find a subsequence of $\{u_{n_k}'\}_k$ that weakly converges to some u'. By renaming of the subsequences, we have found $u_{n_k} \rightharpoonup u \in L^2(\mathbb{R})$ and $u_{n_k}' \rightharpoonup u' \in L^2(\mathbb{R})$. Define $G : L^2(\mathbb{R}) \to [-\infty, \infty]$,

$$G(u) = \int_R |u| dx = ||u||_{L^1}$$

which is convex. Let $C \in \mathbb{R}$ be the bound so that $C \geq ||u_n||_V$ and compute

$$C \geq \liminf_{k\to\infty} ||u_{n_k}||_V \geq \liminf_{k\to\infty} ||u_{n_k}||_{L^1} = \liminf_{k\to\infty} G(u_{n_k}) \geq G(u) = ||u||_{L^1}$$

applying Proposition 16 with $\hat{F} = G$. This ensures that $||u||_{L^1} < \infty$. In combination with $||u'||_{L^2} < \infty$, this tells us that $u \in V$. Finally compute,

$$\inf_V F = \lim_{n\to\infty} F(u_n) = \lim_{k\to\infty} F(u_{n_k}) = \liminf_{k\to\infty} F(u_{n_k}) \geq F(u) \geq \inf_V F$$

applying Proposition 16 with $\hat{F} = F$ and using the continuity of F proved in Section 3.1. This tells us that $F(u) = \inf_V F$ giving the requisite minimizer u. Since F is strictly convex, the minimizer is unique. ∎

3 Continuity and Subdifferentiability

3.1 Continuity

The special choice of the function space V plays an important role in the continuity of F.

Proposition 17. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. Then, F is continuous at $u \equiv 0$.*

Proof. Consider a sequence $u_n \to 0$ in the sense of the norm. That is,

$$\lim_{n\to\infty} ||u_n||_V = \lim_{n\to\infty} ||u_n||_{L^1} + ||u_n'||_{L^2} = 0.$$

Because both terms are nonnegative, we can say that $||u_n||_{L^1} \to 0$ and $||u_n'||_{L^2} \to 0$ as $n \to \infty$. As a consequence, $||u_n'||_{L^2}^2 \to 0$ as well. Compute in three parts

$$\int_{\mathbb{R}} \frac{1}{2}(u_n')^2 dx = \frac{1}{2}||u_n'||_{L^2}^2 \to 0$$

and

$$\left|\int_{\mathbb{R}} f u_n dx\right| \leq \int_{\mathbb{R}} |f u_n| dx \leq ||f||_{L^2}||u_n||_{L^2} \leq ||f||_{L^2}(||u_n||_{L^1} + ||u_n'||_{L^2}) \to 0$$

using Hölder inequality and inequality (3) and

$$\int_{\mathbb{R}} \frac{1}{\mu}|u_n| dx = \frac{1}{\mu}||u_n||_{L^1} \to 0.$$

So, we conclude

$$\begin{aligned}\lim_{n\to\infty} |F(u_n)| &= \lim_{n\to\infty} \left|\int_{\mathbb{R}} \frac{1}{2}(u_n')^2 - f u_n + \frac{1}{\mu}|u_n| dx\right| \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}} \left|\frac{1}{2}(u_n')^2 - f u_n + \frac{1}{\mu}|u_n|\right| dx \\ &\leq \lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{2}(u_n')^2 + |f u_n| + \frac{1}{\mu}|u_n| dx = 0.\end{aligned}$$

So indeed we have $\lim_{n\to\infty} F(u_n) = 0 = F(u)$ as desired. ∎

It is then possible to prove that F is continuous on V by appeal to Theorem 4.43 [4, p. 273] given here

Theorem 18. *Let V be a locally convex topological vector space and let $g : V \to [-\infty, \infty]$ be convex. Then the following are equivalent:*

1. *g is a proper function and is continuous in the interior of the effective domain, which is nonempty;*
2. *there exists a nonempty open set on which g is not identically $-\infty$ and is bounded from above.*

3.2 Subdifferentiability

As noted before, we cannot analyze minimizers using the Euler-Lagrange equation from calculus of variations because the lagrangian is not C^1. But because F is convex, we can use a slightly more general tool for convex functions called the subdifferential. The following definitions and theorems are from Fonseca and Leoni [4, pp. 273-277]. First, we define a subdifferential,

Definition 19. *Let V be a locally convex topological vector space, let $g : V \to [-\infty, \infty]$, and let $v_0 \in V$ be such that $g(v_0) \in \mathbb{R}$. The function g is said to be subdifferentiable if there exists $L : V \to \mathbb{R}$ linear and continuous such that*

$$g(v) \geq g(v_0) + L(v - v_0)$$

for all $v \in V$. The function L is called the subgradient of g at v_0 and the set of all subgradients at v_0 is called the subdifferential of g at v_0 and is denoted by $\partial g(v_0)$. Precisely,

$$\partial g(v_0) = \{L \in V^* : g(v) \geq g(v_0) + L(v - v_0) \ \text{ for all } v \in V\}$$

where V^* denotes the dual space of V. If g is not subdifferentiable at v_0, then $\partial g(v_0) := \emptyset$.

One can think of the subgradient as a hyperplane passing through $(v_0, g(v_0))$ that lies "below" the graph of g. Of importance is that if v_0 is a minimizer of g if and only if $0 \in \partial g(v_0)$. Here, we also define the one-sided directional derivative:

Definition 20. *The one-sided directional derivative of g at v_0 in the direction $v \in V$ is defined by*

$$\frac{\partial^+ g}{\partial v}(v_0) := \lim_{t \to 0^+} \frac{g(v_0 + tv) - g(v_0)}{t}$$

This next theorem (Theorem 4.51 in [4]) is important in proving subdifferentiability of F

Theorem 21. *Let V be a locally convex topological space and let $g : V \to [-\infty, \infty]$ be a convex function. If there is $v_0 \in V$ such that $g(v_0) \in \mathbb{R}$ and g is continuous at v_0, then $\partial g(v) \neq \emptyset$ for all v in the interior of the effective domain of g.*

The proof of this theorem uses Hahn-Banach's Theorem and also the continuity result from before. In applying this result with $g = F$ and $v_0 = u_0 \equiv 0$, we find subdifferentiability of F everywhere in the interior of the effective domain. Because $-\infty < F(u) < \infty$, the effective domain of F is V, and the interior of the effective domain is V. Thus, F is subdifferentiable everywhere.

4 Euler-Lagrange Equation

In this section, we "compute" the subdifferential of F and derive the analogy to the weak Euler-Lagrange equation for this problem. After, we can use this to derive the analogy to the strong Euler-Lagrange equation.

4.1 Computing the Subdifferential

For simplicity, define

$$F_1(u) = \int_{\mathbb{R}} \frac{1}{2}(u')^2 dx, \quad F_2(u) = -\int_{\mathbb{R}} fudx, \quad \text{and} \quad F_3(u) = \int_{\mathbb{R}} \frac{1}{\mu}|u|dx$$

so that $F = F_1 + F_2 + F_3$. The main tool used to compute the subdifferential is Proposition 4.58 [4, p. 282] presented here

Proposition 22. *Let V be a locally convex topological vector space and let $g : V \to (-\infty, \infty]$ be a proper convex function. If $v_0 \in V$ is such that $g(v_0) \in \mathbb{R}$ and g is continuous at v_0, then*

$$\frac{\partial^+ g}{\partial v}(v_0) = \max_{L \in \partial g(v_0)} L(v)$$

for all $v \in V$

So, if we are interested in the maximum of the subgradients evaluated in direction $v \in V$ at $u \in V$ we look at the one-sided directional derivative. Explicitly,

$$\max_{L \in \partial F(u)} L(v) = \frac{\partial^+ F}{\partial v}(u) = \frac{\partial^+ F_1}{\partial v}(u) + \frac{\partial^+ F_2}{\partial v}(u) + \frac{\partial^+ F_3}{\partial v}(u) \tag{4}$$

assuming a priori that the individual directional derivatives exist. We begin the computations thusly. Consider a point $u \in V$ and a direction $v \in V$ and compute

$$\frac{\partial^+ F_1}{\partial v}(u) = \lim_{t \to 0^+} \frac{1}{2t} \int_{\mathbb{R}} ((u + tv)')^2 - (u')^2 dx = \int_{\mathbb{R}} u'v'dx$$

and

$$\frac{\partial^+ F_2}{\partial v}(u) = \lim_{t \to 0^+} \frac{1}{t} \int_{\mathbb{R}} -f(u + tv) + fudx = \int_{\mathbb{R}} -fvdx.$$

The one-sided directional derivative for F_3 is more complicated.

$$\begin{aligned} \frac{\partial^+ F_3}{\partial v}(u) &= \lim_{t \to 0^+} \frac{F_3(u + tv) - F_3(u)}{t} \\ &= \frac{1}{\mu} \lim_{t \to 0^+} \int_{\mathbb{R}} \frac{|u(x) + tv(x)| - |u(x)|}{t} dx \end{aligned}$$

Considering that absolute value is Lipschitz with Lipschitz constant 1, we know that

$$\left| \frac{|u(x) + tv(x)| - |u(x)|}{t} \right| \leq \frac{1}{t} t|v(x)| = |v(x)|$$

which is a measurable function because $v \in V \implies ||v||_{L^1} < \infty$. Thus, we can apply Lebesgue dominated convergence to get

$$\begin{aligned}\frac{\partial^+ F_3}{\partial v}(u) &= \frac{1}{\mu}\lim_{t\to 0^+}\int_{\mathbb{R}} \frac{|u(x)+tv(x)|-|u(x)|}{t}dx \\ &= \frac{1}{\mu}\int_{\mathbb{R}} \lim_{t\to 0^+}\frac{|u(x)+tv(x)|-|u(x)|}{t}dx.\end{aligned}$$

So, we are interested in the value inside the integral in terms of u and v. We compute

$$\begin{aligned}\lim_{t\to 0^+}\frac{|u(x)+tv(x)|-|u(x)|}{t} &= \begin{cases} v(x) & \text{if } u(x)>0 \\ -v(x) & \text{if } u(x)<0 \\ |v(x)| & \text{if } u(x)=0\end{cases} \\ &= \operatorname{sgn}(u(x))v(x) + \mathbb{1}_{u(x)=0}|v(x)|,\end{aligned}$$

giving

$$\frac{\partial^+ F_3}{\partial v}(u) = \frac{1}{\mu}\int_{\mathbb{R}} \operatorname{sgn}(u(x))v(x) + \mathbb{1}_{u(x)=0}|v(x)|dx.$$

Plugging into equation (4) gives

$$\max_{L\in\partial F(u)} L(v) = \int_{\mathbb{R}} u'v' - fv + \frac{1}{\mu}\left(\operatorname{sgn}(u)\cdot v + \mathbb{1}_{u=0}|v|\right)dx \tag{5}$$

abbreviating $\mathbb{1}_{u(x)=0}$ as $\mathbb{1}_{u=0}$. Computing the minimum is accomplished with a simple trick

$$\min_{L\in\partial F(u)} L(v) = -\max_{L\in\partial F(u)} -L(v) = -\max_{L\in\partial F(u)} L(-v)$$

using the linearity of L. This gives

$$\begin{aligned}\min_{L\in\partial F(u)} L(v) &= -\int_{\mathbb{R}} u'\cdot(-v)' - f\cdot(-v) + \frac{1}{\mu}\left(\operatorname{sgn}(u)\cdot(-v) + \mathbb{1}_{u=0}|(-v)|\right)dx \\ &= \int_{\mathbb{R}} u'v' - fv + \frac{1}{\mu}\left(\operatorname{sgn}(u)\cdot v - \mathbb{1}_{u=0}|v|\right)dx\end{aligned} \tag{6}$$

by distributing the negative sign.

4.2 The Weak Euler-Lagrange

Except where noted, let u be a minimizer of F. Then, as noted above we have that $0 \in \partial F(u)$. For clarity, this continuous linear will be denoted by L_0. Then, take any direction $v \in V$ and note that

$$0 = L_0(v) \le \max_{L\in\partial F(u)} L(v) = \int_{\mathbb{R}} u'v' - fv + \frac{1}{\mu}\left(\operatorname{sgn}(u)\cdot v + \mathbb{1}_{u=0}|v|\right)dx$$

using equation (5). Similarly, in the other direction

$$0 = L_0(v) \ge \min_{L\in\partial F(u)} L(v) = \int_{\mathbb{R}} u'v' - fv + \frac{1}{\mu}\left(\operatorname{sgn}(u)\cdot v - \mathbb{1}_{u=0}|v|\right)dx$$

using equation (6). These two inequalities can be combined and simplified by subtracting like terms to get

$$-\int_{\mathbb{R}} \mathbb{1}_{\{u=0\}}|v|dx \leq -\int_{\mathbb{R}} u'v' - fv + \frac{1}{\mu}\operatorname{sgn}(u)\cdot vdx \leq \int_{\mathbb{R}} \mathbb{1}_{\{u=0\}}|v|dx$$

or equivalently

$$-\int_{\{u=0\}} |v|dx \leq \int_{\mathbb{R}} u'v' + \left(-f + \frac{1}{\mu}\operatorname{sgn}(u)\right) vdx \leq \int_{\{u=0\}} |v|dx \tag{7}$$

for all $v \in V$. This can be treated as an analogue of the weak Euler-Lagrange in the C^1 case.

4.3 The Strong Euler-Lagrange

Let $P = u^{-1}((0,\infty))$ and $N = u^{-1}((-\infty,0))$. As u is absolutely continuous, we have that P and N are open subsets of $\mathbb{R}$. Take an interval $(a,b) \subseteq P$. In this interval, $\operatorname{sgn}(u) = 1$. Take any $v \in C_c^\infty((a,b))$ and extend it to be 0 elsewhere giving $v \in V$. The function v is only nonzero on $(a,b) \subseteq P$ where $u(x) > 0$. Thus, $\int_{\{u=0\}} |v(x)|dx = 0$. Plugging into equation (7), we derive

$$0 \leq \int_{\mathbb{R}} u'(x)v'(x) + \left(-f(x) + \frac{1}{\mu}\right) v(x)dx \leq 0.$$

The support of the function inside the integral is contained within $[a,b]$, so we can write

$$\int_a^b u'(x)v'(x) + \left(-f(x) + \frac{1}{\mu}\right) v(x)dx = 0$$

which is true for all $v \in C_c^\infty((a,b))$. We now present a general form of the fundamental lemma of the calculus of variations from Giaquinta and Hildebrandt [5, p. 32].

Lemma 23. *Suppose that Ω is a domain in $\mathbb{R}^n$ and that $g \in L^1(\Omega)$ satisfies*

$$\int_\Omega g(x)\frac{\partial \eta}{\partial x_i}(x)dx = 0, \quad 1 \leq i \leq n$$

for all $\eta \in C_c^\infty(\Omega)$. Then g coincides with a constant function $\mathcal{L}^n$ almost everywhere on Ω.

This leads us to a useful form of the fundamental lemma here

Corollary 24. *Let $g, h \in L^1((a,b))$ and suppose*

$$\int_a^b [g(x)\phi(x) + h(x)\phi'(x)]dx = 0$$

holds for each $\phi \in C_c^\infty((a,b))$. Then the function h is absolutely continuous on $[a,b]$ and $h'(x) = g(x)$ almost everywhere in $[a,b]$.

Proof. Define $G(x) := \int_a^x g(t)dt$ and integrate by parts to get

$$\begin{aligned}
0 &= \int_a^b [g(x)\phi(x) + h(x)\phi'(x)]dx \\
&= G(a)\phi(a) - G(b)\phi(b) + \int_b^a -G(x)\phi'(x) + h(x)\phi'(x)dx \\
&= 0 + \int_b^a (-G(x) + h(x))\phi'(x)dx
\end{aligned}$$

for all $\phi \in C_c^\infty((a,b))$. Noting that $-G + h \in L^1((a,b))$, applying the previous lemma with $\Omega = (a,b)$ gives

$$-G(x) + h(x) = C \implies h(x) = C + G(x) = C + \int_a^x g(t)dt$$

$\mathcal{L}^1$ almost everywhere in $[a,b]$ for some constant C. Redefining $h(x) := C + G(x)$ (which differs from the "real" h only on a set of Lebesgue measure 0) shows that h is absolutely continuous and $h' = g$, $\mathcal{L}^1$ almost everywhere in $[a,b]$. ■

This is a slight abuse of notation as we have actually replaced h with a function that coincides almost everywhere. This would be an issue later as we cannot reason about the value of h at a single point from its original definition, but this can be fixed with the conditions we have. We want to set $g = -f + \frac{1}{\mu}$ and $h = u'$. The easiest way to show these functions work is the following computation by Hölder's inequality:

$$\int_a^b |g|dx = \int_a^b 1 \cdot |g|dx \leq \left(\int_a^b 1^2 dx\right)^{\frac{1}{2}} \left(\int_a^b g^2 dx\right)^{\frac{1}{2}}$$

which shows that $L^2((a,b)) \subseteq L^1((a,b))$. We know that $-f \in L^2(\mathbb{R})$, so $-f \in L^2((a,b))$ and thus $-f + \frac{1}{\mu} \in L^2((a,b)) \subseteq L^1((a,b))$. Similarly, $u' \in L^2(\mathbb{R})$ so $u' \in L^2((a,b)) \subseteq L^1((a,b))$. Having satisfied the conditions for the corollary, we find that u' is absolutely continuous in $[a,b]$ so that u'' exists almost everywhere in $[a,b]$ and

$$u'' = -f + \frac{1}{\mu} \tag{8}$$

almost everywhere in $[a,b]$ and by extension almost everywhere in P. In N, we have that $\operatorname{sgn}(u) = -1$. Carrying the same calculations through, we find that

$$u'' = -f - \frac{1}{\mu} \tag{9}$$

almost everywhere in N. The following computation gives a regularity result which simplifies further proofs greatly. Let w be the function that coincides with u' $\mathcal{L}^1$ almost everywhere and is absolutely continuous. Let $(a,b) \subseteq P$ and choose $x_0 \in [a,b]$. For every $x \in (a,b)$,

$x \neq x_0$ we can compute by absolute continuity

$$\begin{aligned} u(x) &= u(x_0) + \int_{x_0}^{x} w(t)dt \\ &= u(x_0) + \int_{x_0}^{x} w(x_0) + \left(\int_{x_0}^{t} u''(s)ds \right) dt \\ &= u(x_0) + w(x_0)(x - x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} -f(s) + \frac{1}{\mu} ds \right) dt. \end{aligned}$$

Defining $G(t) = \int_{x_0}^{t} -f(s) + \frac{1}{\mu}ds$, we have that G is absolutely continuous and thus continuous on $[a, b]$. Rearranging, we get

$$\begin{aligned} \frac{u(x) - u(x_0)}{x - x_0} &= w(x_0) + \frac{1}{x - x_0} \int_{x_0}^{x} G(t)dt \\ &= w(x_0) + G(t_c) \end{aligned}$$

for some t_c between x, x_0 by mean value theorem. Take an arbitrary sequence $x_n \to x_0$ so that $t_c \to x_0$ and note that

$$G(t_c) = \int_{x_0}^{t_c} -f(s) + \frac{1}{\mu} ds$$

where $\left| -f(s) + \frac{1}{\mu} \right|$ is measurable on $[a, b]$. This allows us to apply Lebesgue dominated convergence to conclude that $\lim_{x_n \to x_0} G(t_c) = 0$ as the pointwise limit is equivalently 0. This gives

$$\lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = \lim_{x_n \to x_0} w(x_0) + G(t_c) = w(x_0)$$

for all sequences $x_n \to x_0$. Because the limit exists for all sequences converging to x_0, we conclude that

$$u'(x_0) = w(x_0)$$

which is true for all $x_0 \in [a, b]$. The LHS is the true derivative of u using difference quotients, and the RHS is the replacement function w that agreed almost everywhere. What this tells us is that there is no need for replacement at all. Furthermore, the same calculation works in N. Thus, we have the following regularity result.

Proposition 25 (Regularity)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$. Let $u \in V$ be a minimizer of F. Let $(a, b) \subseteq P$ or $(a, b) \subseteq N$ be a maximal interval. u' exists at every point and is absolutely continuous in $[a, b]$ using the one-sided derivatives at a and b.*

Remark 26. *This fixes the problem raised before where we could not reason about u' at a given point. Using this proposition, it is enough to talk about u' as the limit of the difference quotient.*

Unfortunately, we cannot find any simple conditions on $u^{-1}(\{0\})$. Equations (8) and (9) are the analogues to the strong Euler-Lagrange.

5 Support of the Solution

5.1 Conditions on Compact Support

In this section, we prove the main result, namely, Theorem 5. To do this, we show some intermediary results.

Lemma 27. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. If*

$$\limsup_{x\to\infty} |f(x)| < \frac{1}{\mu} \quad \textit{and} \quad \limsup_{x\to-\infty} |f(x)| < \frac{1}{\mu},$$

then P, N with respect to a minimizer u of F do not contain unbounded intervals.

This lemma can be split into four symmetric lemmas in two directions (unbounded intervals going to ∞ and $-\infty$) and two sets (P and N). For simplicity, we prove one of the lemmas (going to ∞ and P).

Proof. Assume for sake of contradiction that $[a, \infty) \subseteq P$ for some $a \in \mathbb{R}$. Because

$$\limsup_{x\to\infty} |f(x)| < \frac{1}{\mu}$$

there exists $\varepsilon > 0$ and x_0 so that $|f(x)| \leq \frac{1}{\mu} - \varepsilon$ for all $x \geq x_0$. This gives $-f + \frac{1}{\mu} \geq \varepsilon$ for all $x \geq x_0$. Without loss of generality, we can choose this x_0 so that $x_0 \in [a, \infty)$ because it is unbounded to the right. Then, using equation (8) and the regularity result (see Proposition 25) we compute

$$\begin{aligned} u(x) &= u(x_0) + \int_{x_0}^{x} u'(t)dt \\ &= u(x_0) + \int_{x_0}^{x} \left(u'(x_0) + \int_{x_0}^{t} u''(s)ds \right) dt \\ &= u(x_0) + u'(x_0)(x - x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} \left(-f(s) + \frac{1}{\mu} \right) ds \right) dt \\ &\geq u(x_0) + u'(x_0)(x - x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} \varepsilon ds \right) dt \end{aligned}$$

which gives

$$\begin{aligned} u(x) &\geq u(x_0) + u'(x_0)(x - x_0) + \int_{x_0}^{x} \varepsilon(t - x_0)dt \\ &= u(x_0) + u'(x_0)(x - x_0) + \frac{\varepsilon}{2}(x - x_0)^2 \end{aligned}$$

for all $x \geq x_0$. Then, because $[a, \infty)$ is unbounded to the right, we can send $x \to \infty$ to find

$$\lim_{x\to\infty} u(x) \geq \lim_{x\to\infty} u(x_0) + u'(x_0)(x - x_0) + \frac{\varepsilon}{2}(x - x_0)^2 = \infty$$

But this is not possible because $\int_{\mathbb{R}} |u| dx < \infty$, so we have a contradiction. ■

Note that we only used the ($\star$) limsup condition to the right. Under the same condition, it can be similarly shown that N does not contain an unbounded interval going to ∞ using equation (9). Furthermore, the proof works going to the left, i.e., if the limsup condition is satisfied going to $-\infty$, then P, N do not contain unbounded intervals going to $-\infty$. Another lemma controls where intervals can appear in P and N.

Lemma 28. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ and $\mu > 0$. Let u be a minimizer of F. Let $\varepsilon > 0$ and suppose that there is x_0 so that*

$$\sup_{x \geq x_0} |f(x)| \leq \frac{1}{\mu} - \varepsilon.$$

Then, there is no maximal interval $(a, b) \subseteq P$, b possibly infinite, so that $a \geq x_0$.

Proof. Assume for sake of contradiction that there is such a maximal interval $(a, b) \subseteq P$. Then, necessarily $u(a) = 0$ because otherwise we could find a larger interval. In addition, we claim that $u'(a) \geq 0$ because the difference quotient at a is always positive. Lastly, we have $-f(x) + \frac{1}{\mu} \geq \varepsilon$ for all $x > a$. Then, as in the proof of Lemma 27 above we have

$$\begin{aligned} u(x) &= u(a) + u'(a)(x-a) + \int_a^x \left(\int_a^t \left(-f(s) + \frac{1}{\mu} \right) ds \right) dt \\ &\geq u(a) + u'(a)(x-a) + \int_{x_0}^x \left(\int_{x_0}^t \varepsilon ds \right) dt \\ &= u(a) + u'(a)(x-a) + \frac{\varepsilon}{2}(x-a)^2 \\ &= u'(a)(x-a) + \frac{\varepsilon}{2}(x-a)^2 \geq \frac{\varepsilon}{2}(x-a)^2 > 0 \end{aligned}$$

for all $x > a$. So, we have that $b = \infty$. However, this is in contradiction of the previous lemma because

$$\limsup_{x \to \infty} |f(x)| \leq \sup_{x \geq x_0} |f(x)| \leq \frac{1}{\mu} - \varepsilon < \frac{1}{\mu}$$

and so there cannot be an unbounded interval in P. ∎

Note that this proof also works for N using the appropriate inequalities. Further, this works in the $-\infty$ direction as well.

Remark 29. *We note that the condition for the previous lemma is equivalent to the limsup condition. The main difference is that this condition gives an explicit point at which $|f|$ is controlled.*

This prepares us to prove the main result.

Proof of Theorem 5. Because $\limsup_{x \to \infty} |f(x)| < \frac{1}{\mu}$, we can find x_0 and $\varepsilon > 0$ so that

$$\sup_{x \geq x_0} |f(x)| \leq \frac{1}{\mu} - \varepsilon$$

We have three cases to consider: $x_0 \in P$, $x_0 \in N$, and $x_0 \notin P \cup N$.
Case 1: Suppose $x_0 \in P$. Then, we can find a maximal interval $(a, b) \subseteq P$ so that

$a < x_0 < b < \infty$ because we know that the interval cannot be unbounded. Then, we claim that b is a bound on the support of u to the right. Suppose $u(x) \neq 0$ for some $x > b$. Then, we can find some maximal interval $x \in (c, d)$ so that $(c, d) \subseteq P$ or $(c, d) \subseteq N$. Necessarily, $x_0 < b \leq c$, but this contradicts the previous lemma. So, the support of u is bounded by b.
Case 2: The case when $x_0 \in N$ is similar.
Case 3: Suppose $x_0 \notin P \cup N$. Then, x_0 serves as a bound on the support of u. Suppose $u(x) \neq 0$ for some $x > x_0$. Then, we can find some maximal interval $x \in (c, d)$ so that $(c, d) \subseteq P$ or $(c, d) \subseteq N$. Necessarily, $x_0 \leq c$, but this contradicts the previous lemma. So, the support of u is bounded by x_0.

In all cases, we can find a bound on the support of u to the right. Using the conditions to the left, we can analogously find a bound on the support of u to the left. This ensures that the u is compactly supported as desired. ■

Remark 30. *Theorem 5 does not give explicit bounds on the support of u. This is undesirable and indeed can be improved upon.*

5.2 Measure of the Support

In this section, we prove Theorem 7.

Proof of Theorem 7. From equations (8) and (9), we can derive the following

$$1 = \mu(u'' + f)$$

in P and

$$1 = \mu(-u'' - f)$$

in N. Assuming that u is compactly supported, we can compute the measure of the support thusly

$$\begin{aligned}
\mathcal{L}^1(P \cup N) &= \int_P 1dx + \int_N 1dx \\
&= \int_P \mu(u''(x) + f(x))dx + \int_N \mu(-u''(x) + -f(x))dx \\
&= \mu\left(\int_P (u''(x) + f(x))dx + \int_N (-u''(x) + -f(x))dx\right)
\end{aligned}$$

Writing

$$P = \bigcup_{(a_\alpha, b_\alpha) \in I} (a_\alpha, b_\alpha)$$

as a disjoint union of maximal intervals, we can compute

$$\int_P u''(x)dx = \sum_\alpha \int_{a_\alpha}^{b_\alpha} u''(x)dx = \sum_\alpha (u'(b_\alpha) - u'(a_\alpha)) \leq 0$$

because $u'(b_\alpha) \leq 0$ and $u'(a_\alpha) \geq 0$ at the endpoints of maximal intervals. Similarly,

$$\int_N -u''(x)dx \leq 0.$$

So,

$$\mathcal{L}^1(P \cup N) \leq \mu \left(\int_P f(x)dx + \int_N -f(x)dx \right) \leq \mu \int_{P \cup N} |f(x)|dx.$$

This can naively be bounded by $\mu||f||_{L^1}$, but this is not guaranteed to be finite. Instead, using Hölder's inequality gives

$$\begin{aligned} \mathcal{L}^1(P \cup N) &\leq \mu \int_{P \cup N} |f(x)|dx \\ &\leq \mu \left(\int_{P \cup N} 1dx \right)^{\frac{1}{2}} \left(\int_{P \cup N} (f(x))^2 dx \right)^{\frac{1}{2}} \\ &\leq \mu \left(L^1(P \cup N) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (f(x))^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

which when simplified proves Theorem 7. ■

5.3 Absolute Bound on the Support

Using this bound on the measure of the support, we can find an absolute bound on the support of u that depends only on the data of the problem. The result heavily relies on previous results.

Proof of Theorem 8. Assume for sake of contradiction that there is $x \geq x_0 + \mu^2(||f||_{L^2})^2$ so that $u(x) > 0$. Then, let $(a, b) \subseteq P$ be the maximal interval containing x. By Lemma 28, $a < x_0$. Furthermore, $x_0 + \mu^2(||f||_{L^2})^2 \leq x < b$. So,

$$\mathcal{L}^1(P \cup N) \geq \mathcal{L}^1(P) \geq \mathcal{L}^1((a, b)) > \mathcal{L}^1((x_0, x_0 + \mu^2(||f||_{L^2})^2) = \mu^2(||f||_{L^2})^2$$

which is a contradiction of the previous theorem. The rest of the proof is highly similar. ■

This result is good in that the bounds depend only on f, a given. This could be useful when computing numerical solutions.

5.4 Sharpness

The $(\star)$ limsup condition is fairly sharp. This can be seen by constructing a function f with $\limsup_{x \to \infty} |f(x)| = \frac{1}{\mu}$ so that any minimizer of F is not compactly supported. To do this, we first consider the following calculation. Let $a, b, c > 0$ and let $f = a\mathbb{1}_{[0,c]}$ and

$$u(x) = \begin{cases} -bx(x-c) & x \in [0, c], \\ 0 & \text{elsewhere}. \end{cases}$$

This gives $u'(x) = -2bx + bc$ for $x \in (0,c)$ and $u \in V$, so we can calculate

$$\begin{aligned}
F(u) &= \int_0^c \frac{1}{2}(-2bx+bc)^2 + abx(x-c) - \frac{1}{\mu}bx(x-c)dx \\
&= \int_0^c \left(2b^2 + ab - \frac{b}{\mu}\right)x^2 + \left(-2b^2c - abc + \frac{bc}{\mu}\right)x + \frac{b^2c^2}{2}dx \\
&= \left[\frac{1}{3}\left(2b^2 + ab - \frac{b}{\mu}\right)x^3 + \frac{1}{2}\left(-2b^2c - abc + \frac{bc}{\mu}\right)x^2 + \frac{b^2c^2}{2}x\right]_0^c \\
&= \frac{1}{6}bc^3\left(b - a + \frac{1}{\mu}\right).
\end{aligned} \tag{10}$$

Importantly, this calculation only relies on the integral locally on $[0,c]$. Further, with the choice of $a > \frac{1}{\mu}$ and $0 < b < a - \frac{1}{\mu}$, $F(u)$ is negative. With this in mind, we define

$$f = \sum_{n=1}^{\infty}\left(\frac{1}{\mu} + \frac{1}{n}\right)\mathbb{1}_{\left[n, n+\frac{1}{2^n}\right]}.$$

Then, we check that $f \in L^2(\mathbb{R})$.

$$\begin{aligned}
\int_{\mathbb{R}} f^2 dx &= \int_{\mathbb{R}} \sum_{n=1}^{\infty}\left(\frac{1}{\mu} + \frac{1}{n}\right)^2 \mathbb{1}_{\left[n, n+\frac{1}{2^n}\right]}dx \\
&\leq \int_{\mathbb{R}} \sum_{n=1}^{\infty}\left(\frac{1}{\mu} + 1\right)^2 \mathbb{1}_{\left[n, n+\frac{1}{2^n}\right]}dx \\
&= \left(\frac{1}{\mu} + 1\right)^2 \int_{\mathbb{R}} \sum_{n=1}^{\infty} \mathbb{1}_{\left[n, n+\frac{1}{2^n}\right]}dx \\
&= \left(\frac{1}{\mu} + 1\right)^2 \sum_{n=1}^{\infty}\frac{1}{2^n} = \left(\frac{1}{\mu} + 1\right)^2,
\end{aligned}$$

which is finite. And by inspection $\limsup_{x\to\infty}|f(x)| = \frac{1}{\mu}$ as desired. Assume for sake of contradiction that there is u a minimizer of F so that u is compactly supported. Let the support of u be bounded on the right by C. By construction there is some $\left(\frac{1}{\mu} + \frac{1}{n}\right)\mathbb{1}_{\left[n, n+\frac{1}{2^n}\right]}$ with $n > C$. Choose $b = \frac{1}{2n}$ and let

$$v(x) = \begin{cases} -b(x-n)\left(x - \left(n + \frac{1}{2^n}\right)\right) & x \in \left[n, n+\frac{1}{2^n}\right], \\ 0 & \text{elsewhere.} \end{cases}$$

We then compute

$$\begin{aligned}
F(u+v) - F(u) &= \int_n^{n+\frac{1}{2^n}} \frac{1}{2}(v'(x))^2 - f(x)v(x) + \frac{1}{\mu}|v(x)|dx \\
&= \frac{1}{6}\frac{1}{2n}\left(\frac{1}{2^n}\right)^3\left(\frac{1}{2n} - \left(\frac{1}{\mu} + \frac{1}{n}\right) + \frac{1}{\mu}\right) \\
&= \frac{1}{6}\left(\frac{1}{2^n}\right)^3\left(-\frac{1}{2n}\right) < 0
\end{aligned}$$

by carrying out the initial calculation (equation (10)) with $a = \frac{1}{\mu} + \frac{1}{n}$, $b = \frac{1}{2n}$, and $c = \frac{1}{2^n}$ and using translation invariance of the integral. This implies that u was not a minimizer, so any minimizer cannot be compactly supported. A very similar argument/construction shows that if f is continuous and $\limsup_{x\to\infty} |f(x)| > \frac{1}{\mu}$, then minimizers of F are not compactly supported, but this is not particularly illuminating.

6 Conclusion

In this paper, we consider a one dimensional variational problem introduced by Osher and Yin [8] that is suited for computation and numerics. The difficulty in analysis is due to the presence of a non-differentiable term in the functional.

With some basic results in Section 2, we show that this problem is well defined along with convexity of the problem. Using this and tools from functional analysis, we show that minimizers do exist for any given f in Section 2. In the following Section 3, we show that the problem is continuous and in fact subdifferentiable everywhere. This is very powerful as it allows us to find necessary conditions on minimizers. Leveraging this, we compute subdifferentials and find analogues of the weak and strong Euler-Lagrange equations in Section 4. There is a slight loss of power as we cannot find necessary conditions where minimizers u take on a value of 0. However, we can find a fairly strong regularity result on $u^{-1}((0,\infty))$ and $u^{-1}((-\infty,0))$.

Finally, using these computations Section 5.1 presents sufficient conditions for compact support of minimizers in the main result Theorem 5. Further, a bound on the measure of the support is given in Section 5.2, and a bound on the support independent of minimizer u is given in Section 5.3. Lastly, sharpness of the sufficient conditions is given in Section 5.4.

Here we very briefly discuss the applicability and the possible generalizations of techniques and results presented in this paper. The existence of minimizers of F can be extended to a larger class of convex functions. The proof given only uses convexity and a certain bound on the norm given a bound on the function. Continuity of the problem largely depends on function and space considered, but subdifferentiability follows quickly. From here, the subdifferential can be calculated for a large class of functions using one-sided derivatives. In particular for minimization problems, this can give necessary conditions on u.

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