Resilient Jammed Packing: A Novel Feature of a Classic Geometry Problem

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Abstract

This paper introduces a novel packing feature called *resiliency*. A sphere packing is considered resilient if the packing remains jammed even after several spheres are removed. Resilient jammed packings have various applications, such as in shipping, where a resilient jammed packing ensures the safety of the item being shipped even if some of the packing material breaks. In 2D, we prove that (1) a minimum of two disks must be removed to unjam the hexagonal packing, and (2) any other lattice packing can be unjammed after one disk is removed. In 3D, we prove that (1) a minimum of three spheres must be removed to unjam the face-centered cubic packing, and (2) any other lattice packing can be unjammed by removing at most two spheres. These results imply that the hexagonal packing is the most resilient 2D lattice packing and the face-centered cubic packing is the most resilient 3D lattice packing.

1 Introduction

A packing of spheres involves the placement of spheres into the space without overlap. For hundreds of years, some of the greatest scientists of all time, including Newton, Kepler, and Gauss, have all worked on the sphere packing problem. However, many key problems still remain open today, and some major breakthroughs have only been achieved recently. For example, the famous Kepler conjecture states that the face-centered cubic packing (along with an infinite number of other similar configurations, including the hexagonal close packing) is the densest 3D sphere packing (see Figure 1a for the face-centered cubic packing). This centuries old conjecture was only proved in 1998 by T. C. Hales [8], using a computer aided proof.

A typical sphere packing problem involves the study of certain packing properties, such as the packing density or the order metric. One important topic in sphere packing is **jammed packing**, where informally, jammed means that each sphere cannot be moved. Formally, a packing is defined as **locally jammed** if each sphere in the packing cannot be moved while the positions of all other spheres in the packing are fixed [15].

The field of jammed packing is an active area of research today in physics and chemistry since it is directly connected to the understanding of various structures, such as crystals, glasses, liquids, colloidal suspensions, granular and random media, and even some biological systems [7, 12, 15]. Beyond physics and chemistry, the packing problem in general has applications in transportation, packaging, and communication, since an efficient and optimal packing saves valuable material [4, 11]. Recently, extensive research has been done in search of low density jammed packings in order to find a stable structure that uses as little material as possible. So far, the lowest known density of

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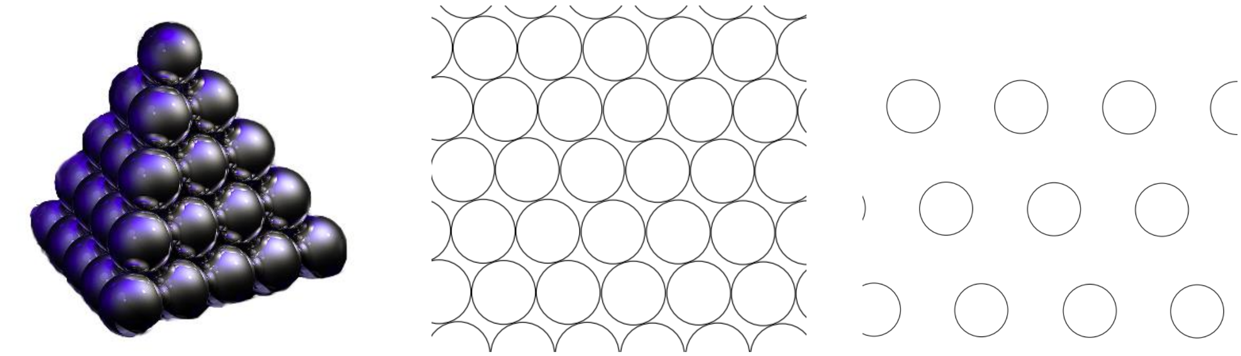

(a) Face-centered cubic packing (b) Jammed lattice packing (c) Non-jammed lattice packing

Figure 1: Lattice packings: in 3D and 2D

a *collectively jammed packing* (a different type of jammed packing where no subset of spheres can be moved simultaneously) in 3D is 49%, which is derived from the 3D honeycomb structure [16]. For mathematical studies on the stability of sphere packings, see [1, 2, 5, 6] for examples.

This paper introduces and studies a novel feature of jammed packings – the **resiliency of a jammed packing**. Generally, a packing is deemed resilient if the *minimum* number of spheres that have to be removed to unjam the packing is large. We define a packing to be **unjammed** if it is not locally jammed. When comparing packings, we will say that packing A is more resilient than packing B if the minimum number of spheres we need to remove to unjam packing A is greater than the minimum number of spheres we need to remove to unjam packing B.

The concept of resiliency has real world applications. For example, in shipping boxes, various packing materials are often placed around the item being shipped in order to protect it. However, fragile items still often break during shipping because the packing material around it is often prone to breaking itself. A packing with a high resiliency would be useful in packaging since it would ensure the safety of the item being shipped, even if some of the packing material breaks. Many shipping companies today use sealed air (plastic containers of air) to fill up empty space in shipping boxes, and these materials are often spherical and can be modeled by sphere packings.

In general, building blocks in packings that are studied in physics and chemistry tend to follow certain repeated patterns similar to lattice packings. A **lattice packing** is defined as a packing of congruent spheres in which the centers of the spheres form a lattice.[1] An example of a jammed lattice packing is shown in Figure 1b. Also note that not all lattice packings are jammed, such as the one shown in Figure 1c. However, we will not study non-jammed lattice packings because they are not resilient.

In this paper, we focus our study on the resiliency of lattice packings in both 2D and 3D. In 2D, we prove that (1) a minimum of two disks must be removed to unjam the hexagonal packing, and (2) any other lattice packing becomes unjammed after just one disk is removed. In 3D, we prove that (1) a minimum of three spheres must be removed to unjam the face-centered cubic packing, and (2) any other lattice packing can be unjammed after two spheres are removed. These results imply that the hexagonal packing is the most resilient 2D lattice packing and the face-centered cubic packing is the most resilient 3D lattice packing.

We would like to point out that establishing the results for 3D is much more involved than for the 2D case. As an example of how difficult it can become to study 3D packing problems in general,

[1] For non-lattice packings, some mathematically intriguing structures have been discovered. For example, the density of locally jammed packings can be arbitrarily close to zero [3, 10].

we would like to refer to the famous "Problem of 13 Spheres." In a recorded debate between David Gregory and Isaac Newton, Newton claimed that it was impossible for one sphere to touch 13 other spheres simultaneously, while Gregory thought that it was possible. In 2D, it is easy to see that the maximum number of disks that can touch a central disk is 6, but Newton's conjecture for the 3D case was only proved in the 1950s (see [14] for the first proof, and [13] for a literature review). We formally state this result below for future reference.

Proposition 1 *It is impossible for one sphere to touch 13 other congruent spheres simultaneously in 3D.*

2 Main Results

As mentioned in the introduction, a **lattice packing** is defined as a packing in which the centers of the spheres form a lattice. In d dimensions, the lattice formed by the centers of the spheres can be described using integer linear combinations of the basis vectors $v_1, v_2, \cdots v_d$, where all of the vectors are linearly independent. That is, the set of points $\{a_1v_1 + a_2v_2 + \cdots a_dv_d, a_i$ is an integer $\forall 1 \leq i \leq d\}$ is precisely the set of sphere centers in the packing. In both the 2D and 3D subsections, each sphere will be referred to by either its center or the coordinates of its center (e.g., $av_1 + bv_2 + cv_3$).

One important aspect of lattice packings that should be emphasized is that any property deduced about one sphere in the packing holds for all spheres in the packing. The reason for this is that any lattice packing remains the same after being transformed, in the sense of both distance and direction, by $a_1v_1 + a_2v_2 + \cdots + a_dv_d$ for any integers a_i $(i = 1, \cdots, d)$. Because of this, we will only analyze the property of the center sphere (i.e., the sphere centered at the origin) in the proof of each theorem. Moreover, any two packings that are equivalent after a congruence transformation will be considered the same.

Since the spheres we study are all congruent, we can assume without loss of generality that **all of the spheres have a radius of $r = 1$ in all of our proofs**. Since the packings are all non-overlapping, the distance between the centers of any 2 spheres is at least 2. If a packing is a candidate for a resilient jammed packing, it must be jammed to begin with. Thus, we will assume in our proofs that **each sphere is locally jammed to begin with**. In addition, all referenced vectors originate from the origin unless stated otherwise. We first make one observation regarding the conditions for local jamming [15].

Observation 1 *In any d-dimensional packing, a sphere is unjammed if and only if it contains a hemisphere (excluding its boundary) with no touching points at all. That is, a d-dimensional packing is locally jammed if and only if each sphere in the packing has at least $d+1$ touching points on its surface that do not all lie in the same hemisphere (including its boundary).*

2.1 Results for 2D

In this section, we will determine the most resilient jammed 2D lattice packing. First, consider a given jammed 2D lattice packing. Note that disks touch the center disk (the disk centered at the origin) in pairs because a disk $av_1 + bv_2$ touches the center disk if and only if $-(av_1 + bv_2)$ also touches the center disk. From Observation 1, we also know that the center disk must touch at least three disks in order for it to be jammed. In addition, it is well known that a disk can only touch a maximum of 6 other congruent disks in 2D (this can be derived from Lemma 1 below). Combining these two restrictions with the fact that disks touch the center disk in pairs, we can conclude that the center disk must touch either 4 or 6 disks.

We will now consider the two cases separately. If the center disk touches 4 disks, then those 4 disks must have centers of the form $a_1v_1+a_2v_2$, $-(a_1v_1+a_2v_2)$, $b_1v_1+b_2v_2$, $-(b_1v_1+b_2v_2)$ since disks touch the center disk in pairs. We know that disks $a_1v_1+a_2v_2$ and $-(a_1v_1+a_2v_2)$ touch the center disk at points $(a_1v_1+a_2v_2)/2$ and $-(a_1v_1+a_2v_2)/2$, respectively, which are diametrically opposite points on the center disk. We will call these two touching points A and B, respectively. Now consider the diameter AB. Since the other two touching points on the center disk ($(b_1v_1+b_2v_2)/2$, $-(b_1v_1+b_2v_2)/2$) are also diametrically opposite, they must lie on opposite sides of diameter AB. This means that if we remove the disk that touches the center disk at a point above AB, then the semicircle that lies above diameter AB must have no touching points on its perimeter (excluding points A and B). Thus, from Observation 1, the center disk must be unjammed, which means that we can unjam any jammed lattice packing in which the center disk touches 4 other disks by removing just one disk.

Next, we will tackle the case in which the center disk (along with all other disks) touches 6 other disks. First, we will introduce the following lemma regarding touching points:

Lemma 1 *Let C_1 be an arbitrary disk with center O_1 in a 2D lattice packing. Then for* **any** *two touching points A and B on C_1, $\angle AO_1B \geq 60°$.*

Proof: Let C_2 and C_3 be the disks that touch C_1 at A and B, respectively (see Figure 2). Let the centers of C_2 and C_3 be O_2 and O_3, respectively. First note that O_1AO_2 and O_1BO_3 are both straight lines, so $\angle AO_1B = \angle O_2O_1O_3$. Thus, we only need to show that $\angle O_2O_1O_3 \geq 60°$. Consider triangle $\Delta O_2O_1O_3$. Since O_2O_1 and O_3O_1 both have a fixed length of 2, $\angle O_2O_1O_3$ is minimized when O_2O_3 is minimized. The minimal length of O_2O_3 is 2, and in this case, triangle $\Delta O_2O_1O_3$ is equilateral, so $\angle O_2O_1O_3 = 60°$. This means that $\angle O_2O_1O_3 \geq 60°$ in general. ■

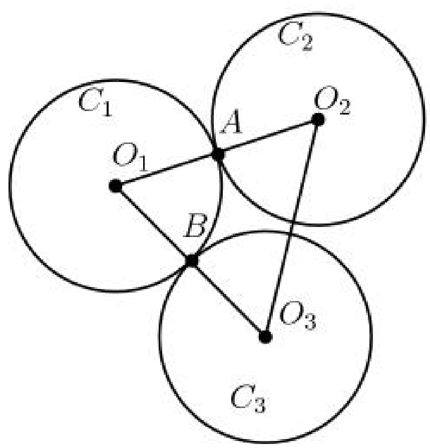

Figure 2: For Lemma 1

Now consider a packing in which every disk touches 6 other disks. Since touching points on a disk must be at least 60° away from each other (Lemma 1) and there are only 360° in a circle, the 6 touching points on every disk must be evenly spaced on the disk's perimeter and form a regular hexagon. This means that the centers of the 6 disks that touch any given disk also form a regular hexagon, so the set of all disk centers must be the hexagonal lattice. Therefore, any packing in which every disk touches 6 other disks must be the hexagonal packing, which is indeed a lattice packing.

We will now show that it is impossible to unjam the hexagonal packing by removing one disk. If one disk is removed, then all disks in the packing must still touch at least 5 other disks. From Lemma 1, we know that any two touching points on a disk must be at least $60°$ apart, which means that it is impossible for the 5 or 6 touching points on any one disk to all lie on the same hemisphere (semicircle in 2D). Thus, based on Observation 1, we can conclude that no disk in the packing can be unjammed.

Finally, we will show that it is possible to unjam the hexagonal packing by removing two disks. From Lemma 1, we know that the 6 touching points on the center disk must be spaced evenly on its perimeter, with $60°$ between adjacent touching points. Thus, if we remove any two disks that touch the center disk at points $60°$ apart from each other, the remaining four touching points on the center disk will all lie in one semicircle, because the furthest remaining touching points are only $180°$ apart from each other. Now Observation 1 implies that the center disk must be unjammed. We have now established our main theorem regarding the resiliency of disk packings in 2D:

Theorem 1 *(1) The hexagonal packing can be unjammed by removing two adjacent disks, but it is impossible to unjam the packing by removing just one disk. (2) Any other 2D lattice packing can be unjammed by removing one disk.*

2.2 Results for 3D

The main goal of this section is to find the most resilient jammed 3D lattice packing. Again, we will only consider whether the center sphere (i.e., the sphere centered at the origin) is unjammed, because any property regarding the center sphere also holds for every single sphere in the packing.

To establish the main theorem, we need to prove a few lemmas, and we begin by establishing the following lemma regarding the basis vectors of a jammed lattice packing.

Lemma 2 *Any jammed 3D lattice packing can be represented using a set of basis vectors v_1, v_2, v_3 with $|v_1| = |v_2| = |v_3| = 2$.*

Proof: First observe that a sphere $av_1 + bv_2 + cv_3$ touches the center sphere if and only if $|av_1 + bv_2 + cv_3| = 2$. Also, note that spheres touch the center sphere in pairs. That is, $av_1 + bv_2 + cv_3$ touches the center sphere if and only if $-(av_1 + bv_2 + cv_3)$ also does.

Now, let the basis vectors of a jammed 3D lattice packing be v_i, v_j, v_k, with $|v_i|, |v_j|, |v_k|$ not necessarily equal to 2. Because the packing is jammed, by Observation 1, the center sphere must have at least 4 touching points on its surface. Therefore, there must exist at least 4 spheres that are a distance of exactly 2 away from the origin. Let two of those spheres be $\pm(a_1v_i + a_2v_j + a_3v_k)$, and the other two spheres be $\pm(b_1v_i + b_2v_j + b_3v_k)$, for some integers a_i and b_i. However, the centers of the 4 spheres listed above all lie in the plane P formed by $a_1v_i + a_2v_j + a_3v_k$, $b_1v_i + b_2v_j + b_3v_k$, and the origin. This means that the center sphere cannot be jammed if those 4 spheres are the only ones touching it (by Observation 1). Thus, there must also exist a third pair of spheres (whose centers are not in plane P) $\pm(c_1v_i + c_2v_j + c_3v_k)$, for some integers c_i, that touch the center sphere.

Now let $v_1 = a_1v_i + a_2v_j + a_3v_k, v_2 = b_1v_i + b_2v_j + b_3v_k, v_3 = c_1v_i + c_2v_j + c_3v_k$, and so $|v_1| = |v_2| = |v_3| = 2$ since each of the spheres touches the center sphere. We will show that the lattice packing G' represented by v_i, v_j, v_k is the same as the lattice packing G represented by v_1, v_2, v_3 by showing that $G \subseteq G'$ and $G' \subseteq G$.

First, note that v_1, v_2, and v_3 are linearly independent since $a_1v_i + a_2v_j + a_3v_k \neq \pm(b_1v_i + b_2v_j + b_3v_k)$ and $c_1v_i + c_2v_j + c_3v_k$ does not lie in plane P. Clearly, every sphere in the packing G is in the packing G', since every integer linear combination of v_1, v_2, v_3 is also an integer linear combination of v_i, v_j, v_k. We now must show that $G' \subseteq G$.

We will prove this by showing that for any point u that is not in the lattice L formed by v_1, v_2, v_3, there exists a point in L that is a distance of less than two away from u. This will show that $G' \subseteq G$ since if there exists a sphere in G' but not in G, then its center (not in L) must be a distance of at least two away from *all* points in L.

Assume for contradiction that there exists a point u not in lattice L that is a distance of at least two away from all points in L. Because v_1, v_2, v_3 are linearly independent, we can write u in the form $u = k_1v_1 + k_2v_2 + k_3v_3$ for some real numbers k_1, k_2, k_3 (not all integers because u is not in L). Without loss of generality, let v_1 and v_2 be in the xy plane and $v_3 = (x_0, y_0, z_0)$ with $z_0 > 0$. Let Q be the plane defined by the equation $z = [k_3]z_0$, where $[k_3]$ denotes the integer closest to k_3 (with 0.5 rounded up). See Figure 3 for an illustration, and note that Q is parallel to the xy plane.

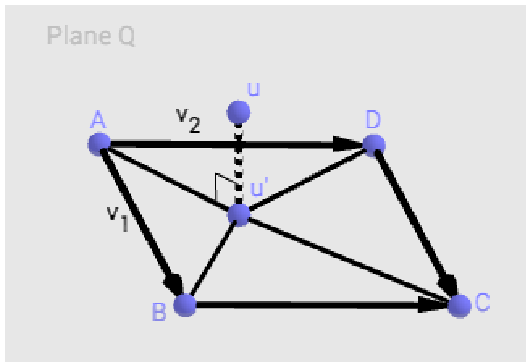

Figure 3: For Lemma 2

Then, we have

$$\text{distance}(u, Q) = |k_3z_0 - [k_3]z_0| \leq \frac{1}{2}|z_0| \leq \frac{1}{2} \cdot 2 = 1, \tag{1}$$

where the last inequality holds because $|v_3| = 2$. Let u' be the projection of u onto plane Q. Note that for any integers a and b, the point $av_1 + bv_2 + [k_3]v_3$ (which is in L) is in plane Q. This means that there exists a lattice point $A = n_1v_1 + n_2v_2 + [k_3]v_3$ for some integers n_1 and n_2 such that u' is contained in the rhombus formed by the points $A, B = A + v_1, C = A + v_1 + v_2, D = A + v_2$, with A, B, C, D all being points in L. Note that the quadrilateral formed by A, B, C, D is a rhombus because $|v_1| = |v_2| = 2$. Since the distance from u to plane Q is at most 1 (Inequality 1) and the distances from u to A, B, C, D must each be at least 2 based on our assumption, the distances from point u' to A, B, C, D must each be at least $\sqrt{3}$ (from the Pythagorean Theorem). Now let $\theta = \max(\angle Au'B, \angle Bu'C, \angle Cu'D, \angle Du'A)$, so $90° \leq \theta \leq 180°$. Then by the law of cosines, one of the lengths AB, BC, CD, DA must be at least

$$\sqrt{\sqrt{3}^2 + \sqrt{3}^2 - 2\sqrt{3}\sqrt{3}\cos(\theta)} \geq \sqrt{6}.$$

However, $AB = BC = CD = DA = 2$ since $|v_1| = |v_2| = 2$, so we have a contradiction. ■

Having proven Lemma 2, we will now classify all future jammed lattice packings by using a set of basis vectors v_1, v_2, v_3 satisfying $|v_1| = |v_2| = |v_3| = 2$.

The most resilient 3D lattice packing is found by determining whether or not a packing is actually jammed after several spheres are removed. Using Observation 1, we can see that in order to determine a packing's resiliency, we must first identify the set of spheres that either touch or may touch the center sphere. That is, we want to find all possible triples (a, b, c) such that it is possible for $av_1 + bv_2 + cv_3$ to touch the center sphere. In order to do so, we need to first introduce two more lemmas.

Lemma 3 *Let v_a be a vector with $|v_a| = 2$, v_b be an arbitrary non-zero vector, v'_a be the projection of v_a onto a plane P containing v_b, and α be the angle between v'_a and v_b. Then $2|v'_a|\cos(\alpha) \leq |v_b|$ if $|v_a - v_b| \geq 2$.*

Proof: Let plane P be the xy plane, and the z-coordinate of v_a be z_a. Then,

$$\begin{aligned} |v'_a|^2 + z_a^2 &= 4 && \text{from } |v_a| = 2, && (2) \\ |v'_a - v_b|^2 + z_a^2 &\geq 4 && \text{from } |v_a - v_b| \geq 2, \text{ and} && (3) \\ |v'_a - v_b|^2 &= |v_b|^2 + |v'_a|^2 - 2|v_b||v'_a|\cos(\alpha) && \text{by law of cosines.} && (4) \end{aligned}$$

Plugging Equations 2 and 4 into Inequality 3 gives $|v_b|^2 - 2|v_b||v'_a|\cos(\alpha) \geq 0$ or $2|v'_a|\cos(\alpha) \leq |v_b|$, as desired. ■

Lemma 4 *Let v_1, v_2, and v_3 be the basis vectors of a lattice packing (not necessarily jammed) such that $|v_i| = 2$ for all i. Then the distance between any one of the vectors and the plane containing the other two vectors is at least $\sqrt{2}$.*

Proof: We will only show that v_3 is a distance of at least $\sqrt{2}$ away from the plane containing vectors v_1 and v_2. Without loss of generality, let $v_1 = (2, 0, 0)$ and $v_2 = (2\cos(\theta), 2\sin(\theta), 0)$, with $60° \leq \theta \leq 120°$ (θ must satisfy these bounds because of Lemma 1). Let the projection of v_3 onto the xy plane be v'_3. Then, without loss of generality, assume that v'_3 lies between v_1 and v_2 (if it does not, we can change one or both of the basis vectors to $-v_1$ or $-v_2$ instead). If it can be shown that $|v'_3| \leq \sqrt{2}$, then the z coordinate of v_3 must be at least $\sqrt{2}$ (since $|v'_3|^2 + z_3^2 = 4$, where z_3 is the z coordinate of v_3), which will conclude the proof.

Let O be the origin, and $\alpha = \min(\angle v'_3Ov_1, \angle v'_3Ov_2)$ (see Figure 4). Then α satisfies $\alpha \leq \theta/2$. Without loss of generality, let v'_3 be closer to v_1 than to v_2 (the other case is symmetrical). Applying Lemma 3 with $v_a = v_3$ and $v_b = v_1$ (note that $|v_3 - v_1| \geq 2$ since spheres v_3 and v_1 cannot overlap), we have

$$2|v'_3|\cos(\alpha) \leq |v_1| = 2. \tag{5}$$

Applying Lemma 3 with $v_a = v_3$ and $v_b = v_1 + v_2$ (again note that $|v_3 - v_1 - v_2| \geq 2$), we have

$$2|v'_3|\cos(\theta/2 - \alpha) \leq |v_1 + v_2| = 4\cos(\theta/2), \tag{6}$$

where the equality holds because $\angle v_1O(v_1 + v_2) = \theta/2$ in the rhombus formed by the vertices $O, v_1, v_2, v_1 + v_2$ (i.e., $v_1 + v_2$ bisects $\angle v_1Ov_2$) and the inequality holds because $\angle v'_3O(v_1 + v_2) = \theta/2 - \alpha$.

Inequalities 5 and 6 imply that

$$|v'_3| \leq \min\left(\frac{1}{\cos(\alpha)}, \frac{2\cos(\theta/2)}{\cos(\theta/2 - \alpha)}\right).$$

We will show that at least one of the terms in the expression on the right-hand-side must be less than or equal to $\sqrt{2}$ at all times by considering two cases based on the value of α.

Case 1: If $\alpha < 45°$, then $|v'_3| \leq 1/\cos(\alpha) < \sqrt{2}$, as claimed.

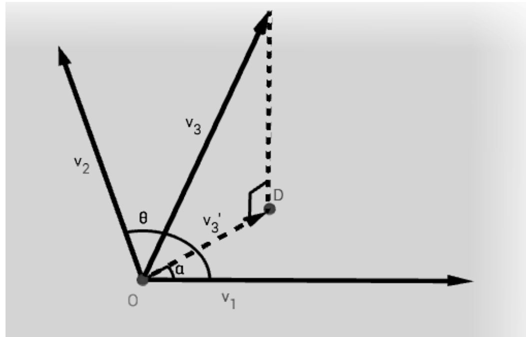

Figure 4: For Lemma 4

Case 2: If $\alpha \geq 45°$, then we need to show that

$$\frac{2\cos(\theta/2)}{\cos(\theta/2-\alpha)} \leq \sqrt{2}.$$

First note that $\theta/2 \geq \alpha$, so we have $90° \leq \theta \leq 120°$ and $\alpha \leq 60°$. We will now look for the maximum of the left-hand-side in the range $90° \leq \theta \leq 120°$, $45° \leq \alpha \leq 60°$. Ignore the constraint $\theta/2 \geq \alpha$ for now, and consider the expression on the left-hand-side for a fixed θ. The expression is maximized when the denominator is minimized, or when $\theta/2 - \alpha$ is as close to $90°$ as possible. Since $0 \leq \theta/2 - \alpha \leq 90°$, this is equivalent to maximizing $\theta/2 - \alpha$, or minimizing α. The minimum of α is $45°$ in the case under study, so the expression becomes $2\cos(\theta/2)/\cos(\theta/2 - 45°)$. If we let $f(x) = 2\cos(x)/\cos(x-45°)$, then $f'(x) = 2\sin(-45°)/\cos^2(x-45°)$. In the given range of θ, $f'(x) < 0$. Thus, the maximum of $2\cos(\theta/2)/\cos(\theta/2-45°)$ is $\sqrt{2}$ when $\theta = 90°$, as desired. ■

We will now limit the possible triples (a, b, c) such that $av_1 + bv_2 + cv_3$ may touch the center sphere. Consider any sphere $av_1 + bv_2 + cv_3$, with at least one of $|a|$, $|b|$, or $|c|$ being an integer strictly greater than 1. Without loss of generality, let us assume that $|a| > 1$. Then, using Lemma 4, we can conclude that the distance between av_1 and the plane containing vectors v_2 and v_3 is at least $2\sqrt{2}$, so the distance between $av_1 + bv_2 + cv_3$ and the plane containing vectors v_2 and v_3 is at least $2\sqrt{2}$. Since the origin lies in the plane containing vectors v_2 and v_3, the distance between $av_1 + bv_2 + cv_3$ and the origin must be at least $2\sqrt{2}$, so $av_1 + bv_2 + cv_3$ cannot touch the center sphere. This means that if $av_1 + bv_2 + cv_3$ touches the center sphere, then $|a|, |b|, |c| \leq 1$. Using $|a|, |b|, |c| \leq 1$, we can now list in the following lemma the spheres that may touch the center sphere. Note that $\pm v_1, \pm v_2, \pm v_3$ always touch the center sphere because $|v_i| = 2$.

Lemma 5 *The only possible spheres that can touch the center sphere are $\pm v_i$ (3 pairs of spheres with $i = 1, 2, 3$), $\pm(v_i - v_j)$ (3 pairs of spheres with $i \neq j$ and $i, j = 1, 2, 3$), $\pm(v_i + v_j)$ (3 pairs of spheres with $i \neq j$ and $i, j = 1, 2, 3$), $\pm(v_i + v_j - v_k)$ (3 pairs of spheres with $i \neq j \neq k$ and $i, j, k = 1, 2, 3$), and $\pm(v_i + v_j + v_k)$ (1 pair of spheres with $i \neq j \neq k$ and $i, j, k = 1, 2, 3$).*

We will now introduce another lemma that further limits the possible pairs of spheres that can touch the center sphere.

Lemma 6 *At most one pair of spheres from $\{\pm(v_i + v_j - v_k), \pm(v_i + v_j + v_k)$, with $i, j, k = 1, 2, 3$ and $i \neq j \neq k\}$ (4 pairs total) can touch the center sphere at once.*

Proof: We will prove the lemma by contradiction through 2 cases.

Case 1: Assume for contradiction that both the pair of spheres $\pm(v_i + v_j - v_k)$ and the pair of spheres $\pm(v_i - v_j + v_k)$ (for some i, j, k) touch the center sphere. Then $|v_i + v_j - v_k| = 2$ and $|v_j - v_i - v_k| = 2$. The first equality implies that the distance between $v_i + v_j$ and v_k is 2, while the second equality implies that the distance between $v_j - v_i$ and v_k is 2. However, since $v_i + v_j$ and $v_j - v_i$ are a distance of 4 apart, any point that is a distance of 2 away from both points must lie on the midpoint of $v_i + v_j$ and $v_j - v_i$, which is v_j. But this means $v_k = v_j$, which is a contradiction since the vectors must be linearly independent.

Case 2: Assume again for contradiction that both the pair of spheres $\pm(v_i + v_j - v_k)$ and the pair of spheres $\pm(v_i + v_j + v_k)$ (for some i, j, k) touch the center sphere. Then the distance between $v_i + v_j$ and v_k is 2, and the distance between $v_i + v_j$ and $-v_k$ is 2. Since v_k and $-v_k$ are a distance of 4 apart, any point that is a distance of 2 away from both points is the midpoint of v_k and $-v_k$, which is 0. This means that $v_i + v_j = 0$, which again is a contradiction. ■

With these lemmas, we are now ready to prove Theorem 2, which identifies the most resilient jammed 3D lattice packing.

Theorem 2 *(1) It is possible to unjam the face-centered cubic packing by removing 3 spheres, but impossible to unjam the packing by removing 2 spheres. (2) Any other lattice packing can be unjammed by removing at most 2 spheres.*

Proof: First, we will show that it is impossible to unjam the face-centered cubic packing, in which the center sphere touches 12 other spheres, by removing 2 spheres. Assume for contradiction that it is possible to unjam the center sphere by removing 2 spheres. By Observation 1, this means that there must exist a hemisphere on the center sphere that contains at most 2 touching points excluding the hemisphere's boundary. By symmetry, there must be at most 2 touching points on the corresponding complement hemisphere, excluding its boundary. This means that there are at least 8 touching points on the boundary of the original hemisphere described above. However, this is impossible, since a disk can only touch a maximum of 6 other disks in 2D.

Second, we will show that it is possible to unjam the face-centered cubic packing by removing 3 spheres. One possible set of basis vectors for the face-centered cubic packing is $v_1 = (2, 0, 0), v_2 = (1, \sqrt{3}, 0), v_3 = (1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3})$. Using this set of basis vectors, it is easy to see that in the face-centered cubic packing, there are 6 spheres that touch the center sphere in the xy plane. Hence, removing all 3 spheres that touch the center sphere at points strictly above the xy plane will unjam the center sphere.

Next, we will prove the 2nd statement of the theorem. By Proposition 1, we know that a sphere may touch at most 12 other spheres in a lattice packing. Moreover, Hales proved in [9] that in any packing of congruent spheres (lattice or not) where each sphere touches exactly 12 spheres, the 12 spheres touching any one sphere have to be arranged in either the face-centered cubic or the hexagonal close packing configuration. Because the properties of each sphere in a lattice packing have to be the same, if each sphere in a lattice packing has 12 touching points, the packing as a whole has to be either the face-centered cubic packing or the hexagonal close packing, but not a combination of both. Moreover, it is well known that the hexagonal close packing is non-lattice. To see this, we note that the centers of the spheres in the hexagonal close packing can be written as $\left(2i + ((j\ +\ k) \pmod 2), \sqrt{3}\left[j + \frac{1}{3}(k \pmod 2)\right], \frac{2\sqrt{6}}{3}k\right)$, where i, j, and k are integers. We can observe that the sphere with center $(0, 0, 0)$, the sphere with center $(1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3})$ (let $i = 0, j = 0, k = 1$), and the sphere with center $(2, 0, \frac{4\sqrt{6}}{3})$ (let $i = 1, j = 0, k = 2$) are all in the packing. If the hexagonal close packing were to be a lattice packing, then the sphere with center $2(1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3})$ would also have to be in the packing, but it overlaps with the sphere centered at $(2, 0, \frac{4\sqrt{6}}{3})$. Therefore, the hexagonal close packing is not a lattice packing.

Thus, we have established that the face-centered cubic packing is the only lattice packing in which the center sphere touches 12 other spheres.[2] This means that in order to prove statement 2, we only need to show that any lattice packing in which the center sphere touches 10 spheres or less can be unjammed by removing 2 spheres (the center sphere cannot touch exactly 11 spheres since 11 is not an even number). We will now consider two cases.

Case 1: The center sphere touches 10 spheres. Since the center sphere always touches $\pm v_i, \pm v_j$, and $\pm v_k$ (with $i \neq j \neq k$ and $i, j, k = 1, 2, 3$), by Lemma 5, it has to touch 2 more pairs of spheres from $\pm(v_i - v_j)$, $\pm(v_i + v_j)$, $\pm(v_i + v_j - v_k)$, and $\pm(v_i + v_j + v_k)$.

From Lemma 6, at most one pair of spheres from $\pm(v_i + v_j - v_k)$ and $\pm(v_i + v_j + v_k)$ can touch the center sphere at one time, so at least one other pair of spheres has to come from $\pm(v_i - v_j)$ or $\pm(v_i + v_j)$. This means that the center sphere has to have at least 6 of its touching points in the plane that contains vectors v_i and v_j, where the possible candidates are in the set of $\{\pm v_i, \pm v_j, \pm(v_i - v_j), \pm(v_i + v_j)\}$. Since the center sphere has a total of 10 touching points on its surface, it must have at most 2 of its touching points strictly above the plane containing vectors v_i and v_j, and at most 2 touching points strictly below the plane. As a result, we can unjam the center sphere by removing the 2 (at most) spheres that touch the center sphere strictly above the plane.

Case 2: The center sphere touches 6 or 8 other spheres. In this case, the center sphere touches at least the 4 spheres $\pm v_i, \pm v_j$ in the plane containing vectors v_i and v_j. This means that there are either 1 or 2 touching points on the center sphere strictly above the plane, and 1 or 2 touching points strictly below the plane. If all spheres that touch the center sphere strictly above the plane (at most 2) are removed, then the center sphere is unjammed. ■

3 Discussion and Future Work

In this paper, we have introduced the concept of resiliency, which is motivated by real world applications such as shipping and packaging. In this context, as undesirable as it may be for some packing material to become "loose," unjamming is even more damaging if the item being shipped becomes loose. Motivated by this consideration, we define a packing to be **globally unjammed** if every single sphere in the packing can be moved (not necessarily all at once). In contrast to local unjamming, where the protected item may or may not be affected, global unjamming is completely unacceptable because everything, including the protected item, can move and therefore be damaged.

As a natural extension to the results derived so far, we have determined the most resilient jammed lattice packing regarding global unjamming for the 2D case. The exact results are stated in the following proposition, and detailed proofs are available upon request.

Proposition 2 *(1) In the hexagonal packing, it is impossible to globally unjam the packing by removing one disk, but removing two adjacent disks globally unjams the packing. (2) In all other 2D lattice packings, removing any one disk globally unjams the packing.*

Note that the most resilient jammed packing in 2D is the same for both local and global unjamming, and the number of disks we need to remove is also the same for both local and global unjamming. However, the latter fact is simply a coincidence. In fact, our proof for global unjamming in 2D is quite different than the one for local unjamming, because we need to carefully show that every disk can move an explicit distance in order to achieve global unjamming.

[2] In an earlier version of this paper, I proved this statement as a lemma, independent of Hales' result. One of the referees pointed out that the lemma is a corollary of Hales' more general result, which considers both lattice and non-lattice packings.

Moreover, for the 3D case, removing 3 mutually tangent spheres can locally unjam the face-centered cubic packing, but we believe that it does not globally unjam the packing. We now present an informal argument as follows:

If we want to unjam the center sphere in the face-centered cubic packing by removing three spheres, we must remove three mutually tangent spheres (call these three spheres A, B, and C) that all touch the center sphere. This statement can be formally justified by considering the definition of the face-centered cubic packing in the coordinate system. Our claim now is that after spheres A, B, and C are removed, only the center sphere will be unjammed. For simplicity, we will remove the center sphere as well and show that no remaining spheres are unjammed. Now assume for the sake of contradiction that there is another sphere S that is unjammed. Clearly, sphere S cannot touch all of the four spheres A, B, C, and the center sphere. This means that there must be a set of three mutually tangent spheres which originally all touched sphere S that were removed. Since there are only four spheres that have been removed in total, there are only four subsets of three removed spheres that exist. First consider the subset of A, B, and C. Because in the face-centered cubic packing the only sphere that originally touched A, B, and C is the center sphere, the removal of A, B, and C cannot be the reason why sphere S is unjammed. Note that there is no sphere located where the reflection of the center sphere across the plane containing the centers of A, B, and C is, because if there were, the packing would be non-lattice. Similarly, the only sphere that touched the set of spheres A, B, and the center sphere is sphere C, the only sphere that touched A, C, and the center sphere is sphere B, and the only sphere that touched B, C, and the center sphere is sphere A. This means the removal of spheres A, B, C, and the center sphere cannot have caused sphere S to become unjammed, which is a contradiction.

Even though the number of spheres we need to remove to globally unjam the face-centered cubic packing is greater than the number needed to locally unjam it, we still conjecture that the face-centered cubic packing is the most resilient when it comes to global unjamming.

Conjecture 1 *The minimum number of spheres that have to be removed to globally unjam the face-centered cubic packing is 5, and any other 3D lattice packing can be globally unjammed by removing fewer spheres.*

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