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Materials from Mathematics

By Richard D. James

Like much of mathematics, the mathematical study of materials begins with Euler [7], or perhaps with Hooke's models of crystals as periodic arrays of balls ([8], Schem. 7). Some readers might know of a more recent historical touchstone, the N – 6 rule of von Neumann and Mullins [11, 16]. Scientists trained on both sides of the increasingly blurred line between mathematics and materials science have been attracted by the striking beauty of microstructure, the extreme nonlinearity, nonconvexity, and even nonexistence exhibited by theories of materials, and the surprising links between the atomic structure of materials and a host of mathematical subjects, including geometry, calculus of variations, partial differential equations, group theory, graph theory, topology, and harmonic analysis. Mathematics is now guiding the discovery of materials using principles that in some cases run counter to accepted beliefs in materials science, and materials are inspiring new mathematics.

One of the most fruitful areas has been the study of phase transformations. There are a myriad of important phase transformations: solid to liquid, crystalline to amorphous, the ordering of atoms on a lattice, diffusional precipitation, and shape-changing transformations between crystalline forms without diffusion. The latter, called *martensitic phase transformations*, are particularly interesting because they can occur quickly. Highly ordered structures like crystals are famous for their "ferro" properties—ferromagnetism, ferroelectricity, ferroelasticity. The strongest materials and superconductors are also ordered materials. Having a phase transformation between two crystals with different ferro (or other) properties means that the material can be made to switch between these properties: in short, multiferroism by phase transformation. Some of the most interesting technological challenges today involve the possible application of these phase transformations to such fields as microelectronics, information storage, energy conversion, robotics, and sensing.

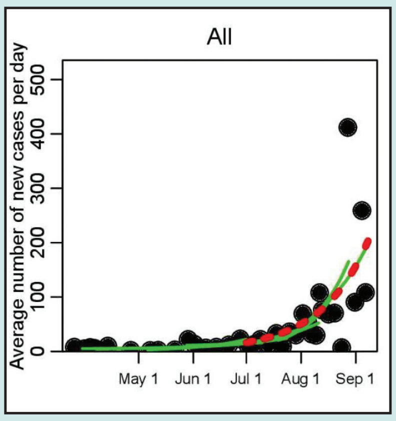
In nearly all of these applications, we want the material to pass back and forth through the phase transformation many times, through heating and cooling. (Ferroic martensitic materials, by the way, can often be made to transform at a fixed temperature, with the application of an electric or magnetic field.) Martensitic materials have a higher transformation temperature on heating than on cooling, a phenomenon known as hysteresis. The loop in a plot of phase-fraction vs. temperature is a *hysteresis loop*. To achieve fast switching of phases, we want a small hysteresis loop, i.e., we do not want to have to heat and cool by hundreds of degrees just to get the material to transform back and forth. Equally problematic for many applications is that the area enclosed by the hysteresis loop is a measure of the energy dissipated by the transformation.

What causes hysteresis? What governs the reversibility of phase transformations? In the pure element tin (Sn), the martensitic phase transformation that occurs around 10º C is so disruptive that during the first cooling cycle the material tears itself apart, yielding a pile of powdered tin. This is often attributed to a large volume change. Other textbook ideas for the origins of hysteresis include the "pinning" of interfaces by defects and the thermally activated crossing of energy barriers.

Mathematical theory suggests a quite different explanation. To understand this,

See **Materials** *on page 5*

Modeling the Dynamics of the Ebola Outbreak



Sherry Towers, Oscar Patterson-Lomba, and Carlos Castillo-Chavez, among those working to model the dynamics underlying the 2014 West African Ebola outbreak, describe their data-driven approach, which "gives public health experts a simple framework that is useful for assessing in near-real time whether control efforts are efficacious."

Shown here, from Figure 1, is a time series of recorded average number of new cases of Ebola virus disease per day (dots) during the initial phase of the 2014 outbreak for all three of the West African countries (Guinea, Sierra Leone, and Liberia) studied.

See page 2.

Periodic Table of the Finite Elements

By Douglas Arnold and
Anders Logg

The finite element method is one of the most powerful and widely applicable techniques for the numerical solution of partial differential equations and, therefore, for the simulation of the physical world. First proposed by engineers in the 1950s as a practical numerical method for predicting the deflection and stress of structural components of aircraft, the method has since been continuously extended and refined. It is now used in almost all application areas modeled by PDEs: solid and fluid dynamics, electromagnetics, biophysics, and even finance, to name just a few.

Finite element methods approximate solutions with piecewise polynomials; the first finite element methods were based on the simplest sorts of piecewise polynomials: continuous piecewise linear functions on triangles, and continuous piecewise bilinear functions on squares. Over the years, to extend the stability, accuracy, and applicability of the method, more complex finite element spaces have been introduced, analyzed, implemented, and applied. In addition to the common and the tensor-product Lagrange elements, these include the serendipity elements, Nédélec elements of various types, the Raviart–Thomas elements, and the Brezzi–Douglas–Marini elements. Even to specialists, the resulting collection can seem a disorganized zoo of possibilities.

Fortunately, much as the chemical elements can be arranged in a periodic table based on their electron structure and recurring chemical properties, a broad assortment of finite elements can be arranged in a table that clarifies their properties and relationships. This arrangement, which is based on expression of the finite element function spaces in the language of differential forms, is one of the major outcomes of the theory known as finite element exterior calculus [4, 5], or FEEC. Just as the arrangement of the chemical elements in a periodic table led to the discovery of new elements, the periodic table of finite elements has not only clarified existing elements but also highlighted holes in our knowledge and led to new families of finite elements suited for certain purposes.

A poster displaying this organized presentation of the principal finite elements is included with this issue of *SIAM News*. It is reduced from the full-size poster, which can be obtained at http://femtable.org. The poster was developed by us with the help of graphic designer Mattias Schläger and the support of Simula Research Laboratory. Harish Narayanan adapted it to the web.

Explanation of the Poster

In our explanation, we assume some familiarity with differential forms on domains in $\mathbb{R}^n$. Readers looking for a brief introduction or review may skip ahead to the final section on differential forms now.

The most prominent aspect of the poster is the arrangement of 108 colored boxes, each corresponding to a finite element space. They form the periodic table of finite elements or, more accurately, a finite section of the table, which is infinite. The element boxes are arranged in four groups, corresponding to the four primary families of finite element spaces, the left-hand two based on simplicial meshes, the right-hand two on cubical or box meshes. The family names, in the notation of FEEC, are $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ for simplices, and $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ for cubes—for short, the $\mathcal{P}^-$, $\mathcal{P}$, $\mathcal{Q}^-$, and $\mathcal{S}$ families.

Each family contains elements for differential forms of all possible degrees k, from 0 up to the space dimension n, as indicated by the Λ^k incorporated into the notation for the family. The form degree determines the coloring, with green boxes used for 0-forms and blue boxes for n-forms, the scalar elements. In 3D, red is used for 1-forms and yellow for 2-forms ($(n - 1)$-forms), the vector elements. In 2D, 1-forms and $(n - 1)$-forms coincide, and are displayed in orange. These correspond to two ways to identify a vector field (v_1, v_2) with a 1-form—as $v_1\, dx^1 + v_2\, dx^2$ or as $v_1\, dx^2 - v_2\, dx^1$—which accounts for the double element diagrams in the orange boxes.

The subscript r in the finite element family name refers to the polynomial degree of the element. The elements exist in any number of space dimensions $n \geq 1$ and for any value of polynomial degree $r \geq 1$, but the poster displays only the lower-order elements (r = 1, 2, and 3) in low dimensions (n = 1, 2, and 3). As we move down the table, the space dimension increases, and for each space dimension the polynomial degree increases. A few elements appear in more than one family and so are repeated on the table.

In summary, each element included in the table corresponds to a choice of

- element family $\mathcal{P}^-$, $\mathcal{P}$, $\mathcal{Q}^-$, or $\mathcal{S}$;
- space dimension $n \geq 1$;
- differential form degree k, with $0 \leq k \leq n$;
- polynomial form degree $r \geq 1$.

For example, the element box in Figure 1, which appears in the 8th row, 6th column of the periodic table, corresponds to the choice $n = 3$, $k = 1$, $r = 2$ in the $\mathcal{P}$ family. This is an element introduced by Nédélec in 1986 [11], commonly called the Nédélec second kind edge element of degree 2. The common name is reflected in the element box in the symbol assigned to the element, in this case $\mathbf{N2}^{\mathbf{e}}_{\mathbf{2}}$. Next to the element symbol in the box appears the notation $\mathcal{P}_2\Lambda^1(\Delta_3)$, which is the FEEC notation for the *shape function* space of the element. Finite element functions are piecewise polynomials, and the shape functions are the polynomial pieces; that is, they are the restrictions of the functions in the global finite element space to a single element. The shape functions for $\mathcal{P}_r\Lambda^k$ are differential k-forms whose coefficients are polynomials of degree at most r. Because a differential 1-form can be viewed as a vector field, the shape function space $\mathcal{P}_2\Lambda^1(\Delta_3)$ consists of vector fields on a tetrahedron Δ_3 for which each of the three components is a polynomial of degree at most 2. The space of such polynomials in 3D has dimension 10 (count the monomials: $1, x, y, z, x^2, xy, \ldots, z^2$), so $\dim\mathcal{P}_2\Lambda^1(\Delta_3)$

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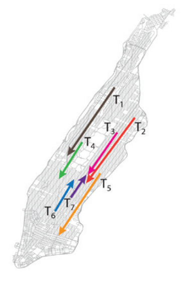
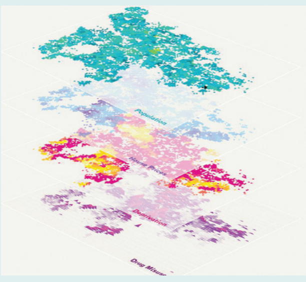


4 ICERM Workshop Sets Out Opportunities and Challenges in Experimental Mathematics
An "exciting" late-July workshop at ICERM "explored emerging challenges of experimental mathematics in the rapidly changing era of modern computer technology." David Bailey and Jonathan Borwein pass along the main findings of the five-day workshop.

12 Opportunities at the Mathematics/Future Cities Interface
Envisioning a central role for mathematical scientists in the emerging interdisciplinary research arena they call "Future Cities," three mathematicians report strong, broad-based support for work in the area and touch on a few of the interesting mathematical challenges they have encountered.



9 Professional Opportunities

Emerging Disease Dynamics

The Case of Ebola

By Sherry Towers,
Oscar Patterson-Lomba, and
Carlos Castillo-Chavez

Sir Ronald Ross introduced the first mathematical model for the transmission of malaria in 1911; this was the de facto creation of the field of mathematical epidemiology as we know it today. Kermack and McKendrick formulated the classic Susceptible, Infected, Recovered (SIR) compartmental model of the spread of disease in 1927. In the ensuing decades these models have been expanded in the broader context of host–parasite dynamics and disease evolution into the robust field of mathematical epidemiology [2, 7].

Soon after September 11, 2001, and the outbreak of Severe Acute Respiratory Syndrome in 2003, modelers across the world mobilized not only to forecast the progression of the SARS outbreak, but also to assess optimal control strategies, including quarantine and isolation (see, for instance, [4]), as well as the threats posed by the deliberate release of biological agents [1]. Identification of the causative agent responsible for SARS led to the quick development of diagnostic tools that, when combined with quarantine and isolation, were ultimately responsible for halting the spread of SARS. It was thus the efforts to assess the potential impact of SARS that highlighted the utility of single-outbreak epidemic models for emerging or re-emerging diseases.

The global health threat posed in 2009 by a pandemic influenza generated by a novel strain of A/H1N1 prompted further theoretical advances in modeling that led to a myriad of immediate and long-term contributions to our understanding of how to best control this global outbreak (see, for instance, [3, 8]). Most of these contributions involved assessing the rate of growth of an epidemic outbreak, estimation of its peak time, and the overall impact (final epidemic size). The models were also used to assess the role that interventions would have in reducing the peak, and to determine the conditions needed to turn a situation of explosive growth into one of no growth or decay.

The dimensionless quantity that plays a key role in assessing all the above factors with SIR Kermack–McKendrick-type models is known as the *basic reproduction number*, or R_0 [2]. R_0 measures the ability of a pathogen to invade a population not previously challenged by a disease, and the average number of secondary infections generated by a typical infectious individual introduced into a purely susceptible population. As time passes, in a closed population with constant transmission rate, the resource (susceptible individuals at time t, $S(t)$) becomes less accessible (because infected or recovered individuals are no longer susceptible). In other words, the rate of growth naturally begins to recede. Hence, the basic reproduction number R_0 is distinguished from the *effective reproduction number*, R_e, and R_e is sometimes modeled by a time-dependent dimensionless quantity $R_0 s(t)$, with $s(t)$ being the proportion of susceptibles in the population at time t. Finding ways of estimating appropriate measures for $R_e(t)$ is critical to assessing the challenges posed by emergent or re-emergent diseases over short times.

The West African Ebola outbreak has inspired several new modeling analyses, motivated by our desire to contribute to the understanding of the dynamics underlying this emerging global health threat. The 2014 Ebola outbreak is characterized by rapidly changing local and regional dynamics, altered by evolving control measures, patterns of spread from rural to densely populated urban areas [5], and behavioral responses in the population that may either inhibit or facilitate the spread of the disease [6]. Therefore, it is evident that the design of real-time control strategies must include temporal components that capture the unfolding dynamics and the variable transmission rate of Ebola.

Our recent paper "Temporal Variations in the Effective Reproduction Number of the 2014 West Africa Ebola Outbreak" makes use of the limited existing data and novel elementary statistical methods (in this context), in combination with a simple single-epidemic nonlinear dynamic model and its associated R_e, to determine whether the transmission rate of Ebola has been changing over time in West Africa [9]. To this end, piecewise exponential curves were fit to the time series of outbreak data (see Figure 1). This ansatz, combined with a mathematical model, was used to estimate the temporal evolution of the effective reproduction number of the disease, estimating the temporal variations in the average number of secondary cases per infectious case in a population composed of both susceptible and non-susceptible individuals. Typically, depletion of susceptible individuals in a closed population during the course of an outbreak would cause the effective reproduction number to decline over time, with a faster-than-expected drop suggesting that control measures and/or changes in population behaviors are effective in inhibiting the spread of the disease. Accordingly, an increasing R_e would indicate a worsening of the conditions.

Unfortunately, yet not surprisingly, rather than a drop in the effective reproduction number, our study showed evidence that the transmission rate of Ebola in Guinea and Liberia actually rose in early August (see Figure 2). What led to the increased transmission rate is somewhat unclear, as many factors could be responsible. Was it increases in the size of the susceptible population linked to the time when the outbreak spread to densely populated cities? Or was it the military-enforced quarantine of entire regions in West Africa, measures that were put in place with no attempt to limit the spread of disease within the quarantined areas, and that may thus have increased the risk of transmission due to crowding, lack of medical and basic services, and poor sanitation? If the latter, then it is clear

See **Ebola** *on page 3*

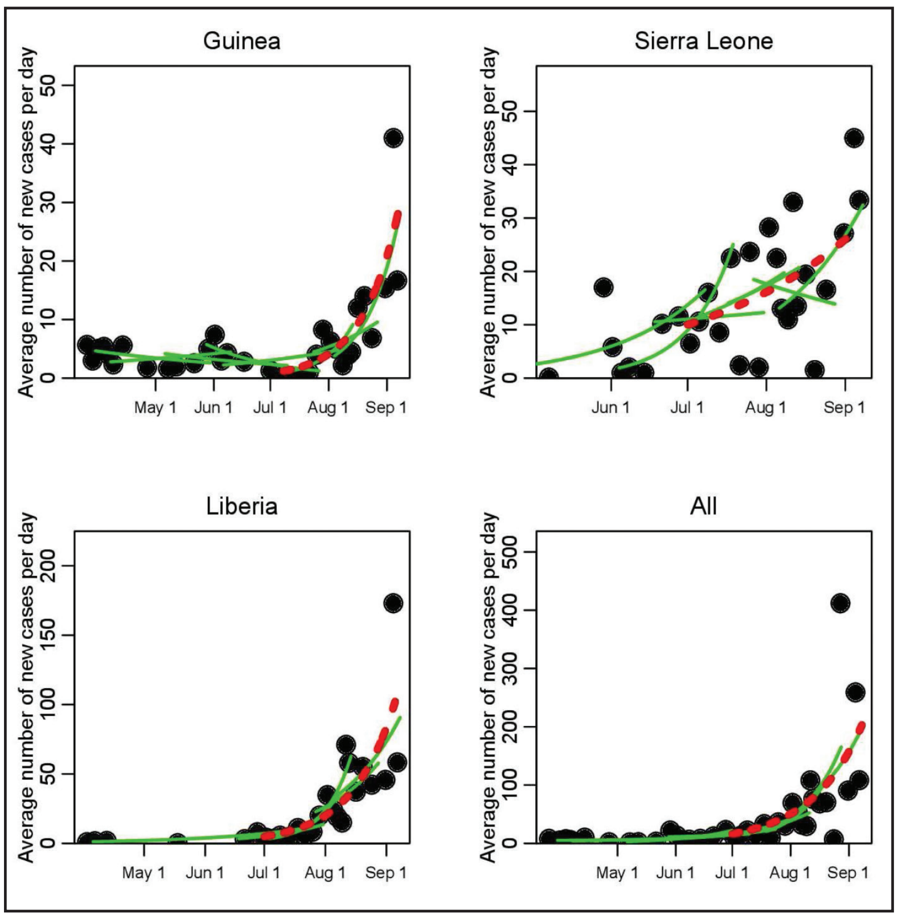


Figure 1. *Time series of recorded average number of new Ebola Virus Disease cases per day during the initial phase of the 2014 West African outbreak, for Guinea, Sierra Leone, and Liberia (dots). The green lines show a selection of the piecewise exponential fits to the data (for clarity of the presentation, not all fits are shown); a moving window takes groups of 10 contiguous points at a time, and the rate of exponential rise (or decline) is estimated for those 10 points. As a reference, the red dotted line shows the fit to all points between July 1 and September 8, 2014. The results for the estimated exponential rise for the full set of piecewise fits are shown in Figure 2.*

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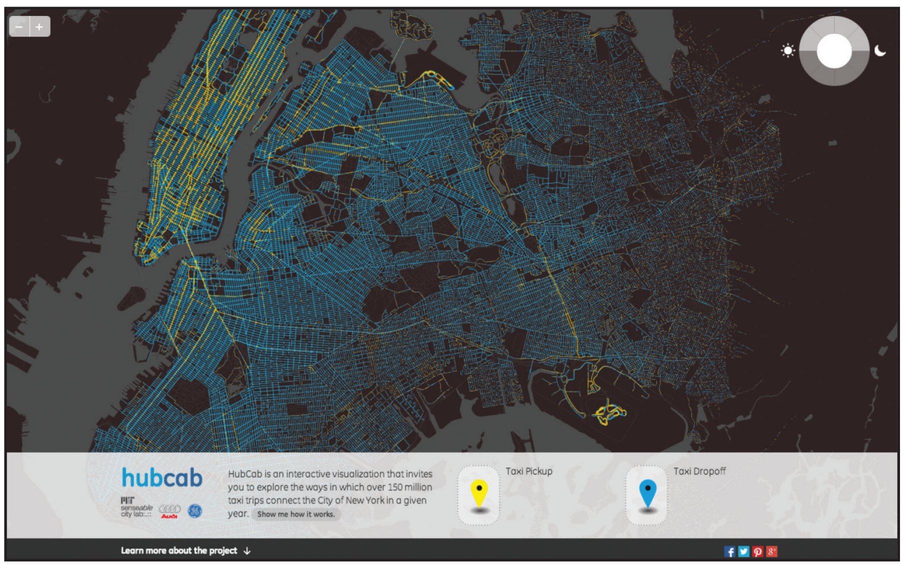
“Shareability Networks”: A New Way to Model the Taxi-sharing Problem

By Paolo Santi

The burgeoning “sharing economy” phenomenon—the collaborative consumption of shared resources made possible by the pervasiveness of information technologies and Internet connectivity [4, 6]—is rapidly taking hold in the context of urban transportation. Business is booming for vehicle-sharing companies, such as Zipcar and Car2Go, and for companies that offer ride-sharing services, such as Bandwagon and Uber, which recently launched the new UberPool application. New sharing services/companies are popping up in cities worldwide.

What are the reasons for this boom? First, urban traffic congestion is a worldwide problem, and it is predicted to become even worse, with an expected tripling in the number of urban trips by 2050 [8]. Second, it is well known that mobility resources are highly underutilized. For instance, most private vehicles lie unused most of the time [3] and typically carry only the driver when in use; in the vast majority of taxi rides, a single passenger is on board [7]. Hence, *sharing* is considered an effective way of increasing the utilization of mobility resources and, consequently, the efficiency of urban traffic: The higher the vehicle-utilization factor, the lower the number of circulating vehicles, which implies less congestion and pollution.

Despite wide agreement on the potential benefits of the shared economy in urban transportation, quantitative, scientifically accurate studies have been lacking. This is due mostly to two factors: the lack of fine-grained spatial and temporal information about urban mobility patterns, and the immense computational and algorithmic challenges of combining massive numbers of trips at the city scale. As to the former, the big data era is opening the way toward an unprecedented understanding of human mobility at the urban scale [2], which is the prerequisite for the task at hand. As to the latter, the recently introduced notion of *shareability networks* [5] is a first example of how suitably defined mathematical models can help tame computational and algorithmic challenges.


hubcab

HubCab is an interactive visualization that invites you to explore the ways in which over 150 million taxi trips connect the City of New York in a given year.

Figure 1. *Visualization of the taxi rides in New York City in 2011, with pick-up locations shown in yellow, drop-off locations in blue. Image courtesy of hubcab.org.*

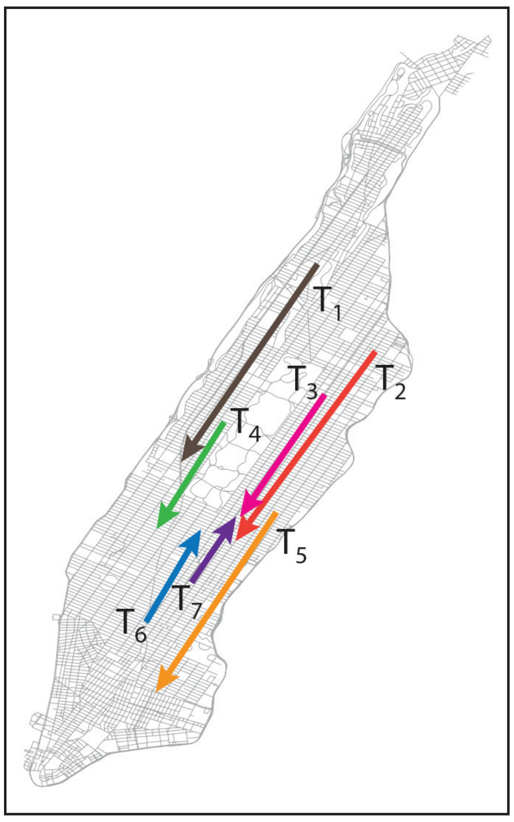
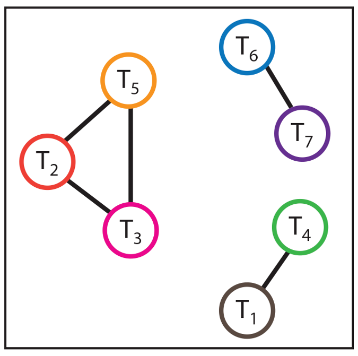


Figure 2. *From taxi trips (left) to “shareability networks” (below).*



In a recently published study [5], we faced the challenge of quantifying the benefits of taxi ride sharing in New York City, starting from a massive data set composed of the more than 150 million trips reported in the city in 2011. A visualization of the trips is available through the HubCab website (hubcab.org) and shown in Figure 1.

Traditionally, ride-sharing problems have been approached as an instance of the general class of dynamic pick-up and delivery problems (DPDP) [1], in which items (taxi passengers, in our case) must be picked up from and delivered to specific locations within well-defined time windows, and the goal is to optimize some criterion, such as total distance traveled or number of vehicles used for pick up/delivery. DPDP are typically solved by means of linear programming, and their computational feasibility depends heavily on the number of variables and equations used to describe the problem at hand. This approach is thus unfeasible when applied to problems like city-wide taxi ride sharing, where the potential number of shared trips (roughly corresponding to the number of system variables) is on the order of several thousands or millions.

In [5], we approached the taxi ride-sharing problem in a novel way. The idea was to use combinatorics to impose a structure on an otherwise unstructured, immense search space, as would be explored in traditional linear programming. To structure the search space, we defined two parameters: the shareability parameter k, the maximum number of trips that can be shared, and the delay parameter Δ, the maximum delay* a customer is willing to tolerate in a shared taxi service. Structuring of the search space, coupled with the notion of shareability networks, as described below, allowed us to find an optimal solution† in an efficient way for an otherwise intractable problem.

Shareability networks are a mathematical model of sharing opportunities. For simplicity, we assume $k = 2$, and our shareability network is a graph of pair-wise sharing opportunities over all trips. We construct the graph by assigning a node to each trip in the data set, and connecting two nodes with an edge if and only if the trips can be shared. Trip shareability is determined on the basis of the existence of at least one route touching the starting and ending points of both trips,‡ such that both passengers arrive at their destinations with delay at most Δ. An example of a taxi trip set and corresponding shareability network is shown in Figure 2.

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*Delay is computed as the difference between the estimated arrival times at the destination in the case of a shared trip and in the case of no sharing (single ride).

†The optimality statement holds subject to the above described constraints.

‡Only routes in which both starting points precede the end points are considered.

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Ebola

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that careful attention must be paid to the design of control measures to ensure that they, first and foremost, do no harm. The analysis further indicated that if the exponential growth in the spread continued, there would have been approximately 4400 new Ebola cases by the beginning of October (95% confidence interval [3000,6800]). Unfortunately, the actual case counts by that date were within the predicted range.

Beyond its applicability to the current Ebola outbreak, this data-driven approach gives public health experts a simple framework that is useful for assessing in near-real time whether control efforts are efficacious.

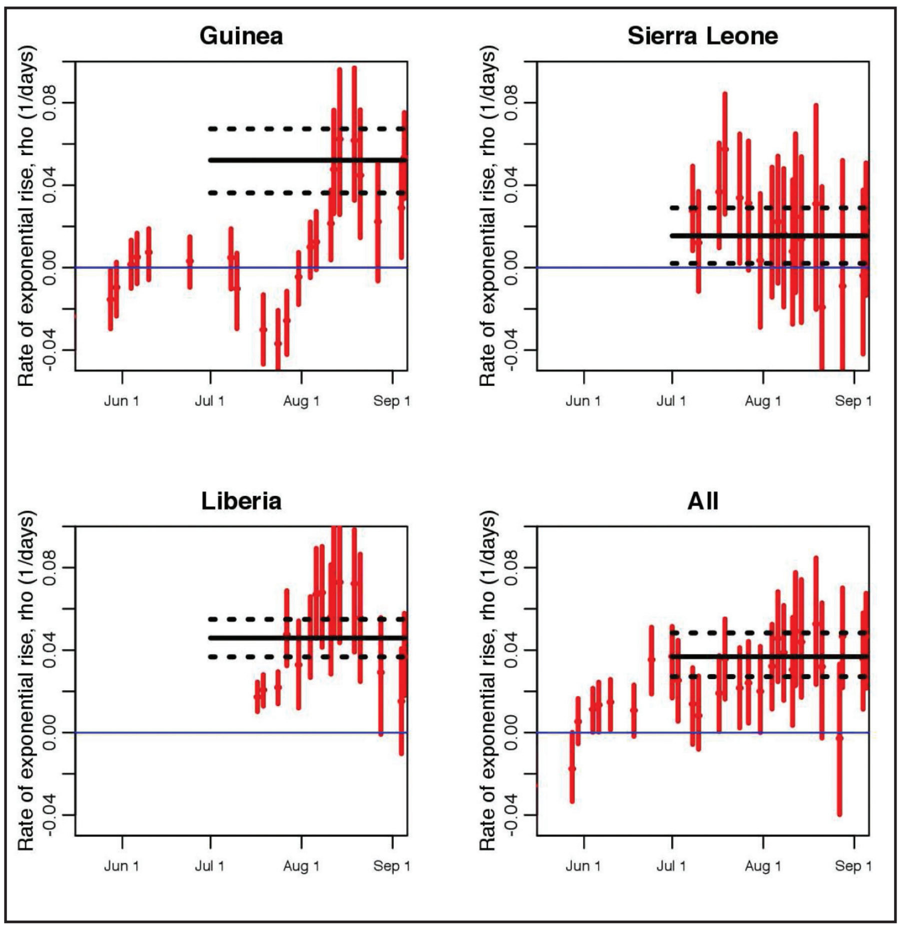

Figure 2. *Estimated exponential rise from piecewise exponential fits to the average daily EVD incidence data, as shown in Figure 1; a moving window takes groups of 10 contiguous incidence data time series points at a time, and the rate of exponential rise is estimated for those 10 points. The dates shown on the x-axis are the last date in each contiguous set of 10 points, and the vertical error bars denote the 95% confidence interval. The horizontal black line shows the estimated rate of rise of an exponential fit to the incidence time series from July 1 to the present, with the black dotted lines indicating the 95% interval.*

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Sherry Towers is a research professor at the Simon A. Levin Mathematical, Computational, and Modeling Sciences Center at Arizona State University. Oscar Patterson-Lomba is a postdoctoral research fellow at the Harvard School of Public Health. Carlos Castillo-Chavez is the Joaquin Buston, Jr. Professor of Mathematical Biology and a Distinguished Sustainability Scientist at Arizona State University.

ICERM Workshop Sets Out Opportunities and Challenges in Experimental Mathematics*

By David H. Bailey and Jonathan M. Borwein

"Experimental mathematics" has emerged in the past 25 years or so as a competing paradigm for research in the mathematical sciences. Challenges in 21st Century Experimental Mathematical Computation, an exciting workshop held at ICERM (the Institute for Computational and Experimental Research in Mathematics), July 21–25, explored emerging challenges of experimental mathematics in the rapidly changing era of modern computer technology. We summarize the workshop findings in this article; information about the research presentations can be found at http://icerm.brown.edu/tw14-5-cemc/.

Despite several more precise definitions that have been offered for "experimental mathematics," we prefer the informal one given in the book *The Computer as Crucible* (Jonathan Borwein and Keith Devlin, AK Peters, 2008):

> "Experimental mathematics is the use of a computer to run computations—sometimes no more than trial-and-error tests—to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search."

"Experimental mathematics" is distinguished from "computational mathematics" and "numerical mathematics" in that the latter two generally encompass methods for applied mathematics, whereas "experimental mathematics" refers to advancing the state of the art in mathematical research per se.

While the overall approach and philosophy of experimental mathematics have not changed greatly in the past 25 years, its techniques, scale, and sociology have changed dramatically. The field has benefited immensely from advances in computer technology, including those predicted by Moore's law, but the increases in speed brought by algorithmic progress have often outpaced Moore's law, notably in such areas as linear programming, linear system solving, and integer factorization.

Software available to experimental mathematicians has also advanced impressively. Along with improvements in earlier versions of commercial products like Maple, *Mathematica*, and MATLAB, many new "freeware" packages are now in use, including the open-source Sage, numerous high-precision computation packages, and an impressive array of software tools and visualization facilities.

With all these tools and facilities, many new results have been published, ranging from new formulas for mathematical constants, such as pi, log(2), and zeta(3), to computer-verified proofs of the Kepler conjecture. Whereas it was once considered atypical or even improper to mention computations in a published paper, now it is commonplace. Several journals, such as *Experimental Mathematics* and *Mathematics of Computation*, are devoted almost exclusively to mathematical research involving computations.

Yet many challenges remain as researchers push the envelope in mathematical computing. Among the most critical issues are the following:

Adapting codes to new platforms. The emergence of powerful, advanced-architecture platforms, particularly those incorporating such features as highly parallel, multi-core, or many-core designs, present daunting challenges to researchers, who must adapt their codes to these new architectural innovations or risk being left behind in the scientific computing world.

Ensuring reliability and reproducibility. Reproducibility means ensuring, for example, that the results of floating-point computations are numerically reproducible, or that the results of a symbolic computation are reliable (complications can arise when two expressions are compared to determine whether they are mathematically equivalent). Many users implicitly trust results obtained with these tools, losing sight of the fact that they are far from infallible. One of the approaches to increased reliability should be stronger interactions with the cousin discipline of formal proof systems (as used by Thomas Hales to complete, in 2014, a multi-year computer-verified proof of the Kepler conjecture on stacking spheres), but huge efficiency issues have to be addressed.

Managing the exploding scale of data. The size of datasets used in the field has increased at least as fast as Moore's law growth. *Algorithmic progress* is thus necessary in, for example, tools that aid in the quest for structure in large numerical or symbolic datasets.

Large-scale software maintenance. The rapidly increasing size of many of the software tools used in the field means that mathematicians must confront the challenge of large-scale software maintenance. This includes the discipline, unfamiliar to many research mathematicians, of strict version control, collaborative protocols for checking out and updating software, validation tests, issues of worldwide distribution and support, and persistence of the code base.

Changing sociological and community issues. Numerous recently published results arise from Internet-based collaborations, with research ideas, computer code, and working manuscripts often circling the globe multiple times in a single day. One example is the PolyMath project, whereby loosely knit Internet-based teams of mathematicians have addressed and, in several cases, "solved" or progressed toward the solution of interesting unsolved mathematical problems. Further progress will require improved tools and platforms for such collaborations, as well as an international "clearing house" that will collect, validate, and coordinate such activities.

Education. Computer-based tools are also being introduced into mathematical education, permitting students to see mathematical concepts emerge from hands-on experimentation and thus attracting to the field a cadre of 21st-century computer-savvy students. This is not the first time that technology has promised to reinvent mathematical education, but it is clear that much additional thought is needed on how computation can be best incorporated into education.

Other issues. The workshop discussion highlighted the fact that much of the published work to date in experimental mathematics has focused on a

See **Experimental Math** *on page 7*

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*The full report, by D.H. Bailey, J.M. Borwein, U. Martin, B. Salvy, and M. Taufer, "Opportunities and Challenges in 21st Century Experimental Mathematical Computation," August 26, 2014, is available at http://www.davidhbailey.com/dhbpapers/icerm-2014.pdf.

Materials

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we focus on the ubiquitous microstructure known as the austenite/martensite interface (Figure 1). During transformation, a lot of individual austenite/martensite interfaces make up the boundary between phases. We can understand its structure in part by solving

$$\inf_{\mathbf{y}} \int_{\Omega} W(\nabla \mathbf{y}(\mathbf{x}), \theta)\, d\mathbf{x}, \tag{1}$$

where $\mathbf{y} : \Omega \rightarrow \mathbb{R}^3$ is a deformation that describes transformation and elastic distortion, θ is the temperature, and the infimum is taken over a suitable finite energy space. W has energy wells, whose precise form comes from careful x-ray diffraction measurements of the crystal structures of the two phases. The austenite/martensite interface is explained as a minimizing sequence of this energy, with $\theta = \theta_c$, the transition temperature. Several features, including the (finite) number of such interfaces, the angles seen in Figure 1, the full 3D structure, and the volume fraction of the bands on the left, are nicely predicted [1, 2].

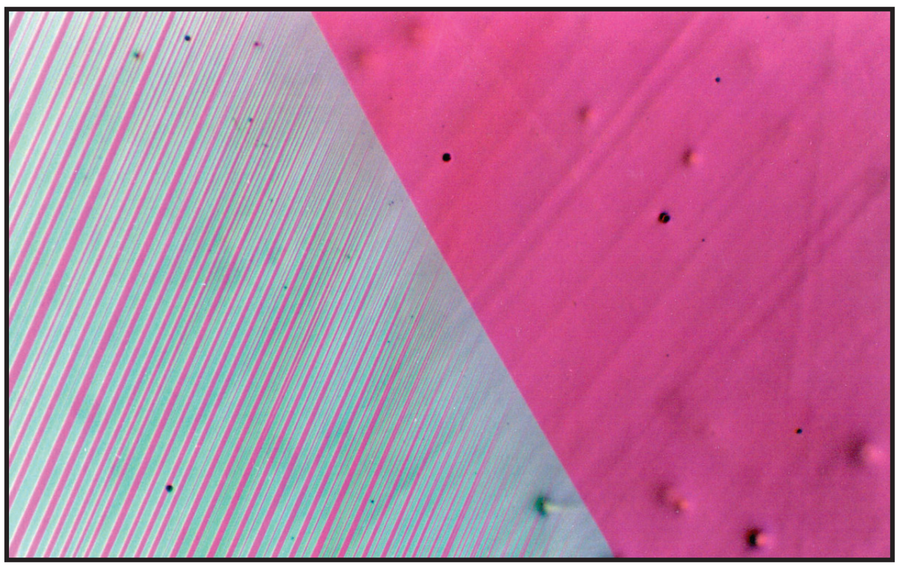

Figure 1. *Austenite/martensite interface. The bands on the left are made up of two variants of martensite; the magenta region at the right is austenite. The total width of the original photograph is about 0.5 mm. Courtesy of C. Chu.*

What is not predicted by this argument is the fineness of the bands on the left. Here again, though, a better mathematical understanding is emerging [4, 5, 10]. The essential idea is that the boundary of each of these bands supports a small interfacial energy per unit area, which is not included in (1). Refining the bands drives the elastic energy in the transition layer between phases (calculated with (1)) to zero, but at the expense of increasing the total interfacial energy. Conversely, coarsening the bands reduces the interfacial energy but gives a big elastic energy. Figure 1 represents the compromise between these two energies. Their sum is a kind of coexistence energy. Whenever both austenite and martensite are present, the material has an additional positive coexistence energy.

But this suggests a reason for hysteresis based on metastability. Suppose that we start in the high-temperature austenite phase and lower the temperature. We reach the temperature at which the two bulk phases have the same free energy, then we lower the temperature a bit more. If martensite appears, we also must accept a (positive) coexistence energy. This will disfavor the transformation to martensite. Mathematically, we should find an energy barrier [18].

The study of this barrier is in its infancy [19], but there is a very simple way to remove it. The energy wells of W have the form $\mathbf{RU}$, where $\mathbf{R}$ is a rotation matrix and $\mathbf{U}$ is a positive-definite symmetric matrix. For the martensite, $\mathbf{U} \in \{\mathbf{U}_1, \ldots \mathbf{U}_n\}$ (determined completely from x-ray measurements[*]), whereas for austenite, $\mathbf{U} = \mathbf{I}$, the identity matrix. We could have an energy minimizer without either the elastic transition layer or the bands on the left of Figure 1, if there were a continuous function $\mathbf{y}(\mathbf{x})$ satisfying

$$\nabla \mathbf{y} = \begin{cases} \mathbf{RU}_1, & \text{for } \mathbf{x} \cdot \mathbf{n} > 0, \\ \mathbf{I}, & \text{for } \mathbf{x} \cdot \mathbf{n} \leq 0 \end{cases} \tag{2}$$

for some 3×3 rotation matrix $\mathbf{R}$. As every undergraduate student in both mathematics and materials science should know [9], (2) holds if and only if $\lambda_2 = 1$, where λ_2 is the middle eigenvalue of $\mathbf{U}_1$. The situation is pictured in Figure 2. When $\lambda_2 = 1$, the phases fit together perfectly. The reason for the complex microstructure in Figure 1 is precisely that $\lambda_2 \neq 1$!

How do we arrange to have $\lambda_2 = 1$? We are given the material, and either $\lambda_2 = 1$ or it does not. But every material has a composition. All its properties, including the value of λ_2, can be modified by compositional changes. This has been done, guided by mathematical theory: New alloys were made, with the value of λ_2 systematically moved closer and closer to 1. The resulting alloy exhibits unprecedented low hysteresis [6, 17]. Earlier, people had made thousands of alloys, even in the systems where $\lambda_2 = 1$ has now been achieved to high accuracy. Why did people not, by accident, hit the composition for which $\lambda_2 = 1$? Hysteresis is so sensitive to λ_2 that, in most cases, they jumped over it. There is a singularity in the graph of the size of the hysteresis vs. λ_2.

This is one way in which mathematics can discover materials: Identify special conditions on material properties at which interesting behavior, particularly singular behavior, is expected, then design compositional changes to achieve those conditions. This is an inverse problem. It can potentially be solved theoretically with first-principles methods, but many properties (including hysteresis) are not currently predictable by those methods. Much remains to be done, and multiscale mathematics is expected to play a central role.

Even stronger conditions of compatibility, called the *cofactor conditions* [3], have been achieved through systematic compositional changes. This recently led [14] to the fascinating alloy $Zn_{45}Au_{30}Cu_{25}$. It shows record low hysteresis for big first-order phase transformations (as low as 0.2° C) and remarkable reversibility. With its changing pattern of microstructure [12] during cyclic transformation, it is unlike any other martensitic material and begs for a dynamic analysis. Satisfaction of the cofactor, or even stronger, conditions in other material systems could lead to revolutionary materials, e.g., a shape-memory material that displaces NiTi, the most popular one (by far), or an oxide material that is able to go back and forth through a ferroelectric transformation many times without cracking.

Ferroic transformations suggest intriguing new applications. Imagine a martensitic alloy with one phase a strong magnet and the other nonmagnetic, and also with $\lambda_2 \approx 1$. If you transform the alloy by, say, heating, the magnetization will suddenly increase. Wrap a coil around the specimen, and, during transformation, a current will be induced in the coil. This is the direct conversion of heat to electricity (i.e., without a separate electrical generator [15]). Mathematically, it involves Maxwell's equations, micromagnetics, thermodynamics, and the theory of phase transformations [13]. Much remains to be understood about this method, and its many ferroic analogs, but it is a promising candidate for recovery of some of the vast heat energy created every day by diverse sources, from a data center to the sun.

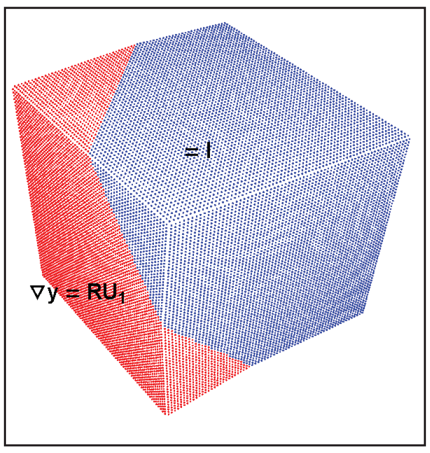

Figure 2. *A perfect austenite/martensite interface is possible if and only if $\lambda_2 = 1$.*

Acknowledgments

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Richard D. James is the Distinguished McKnight University Professor in the Department of Aerospace Engineering and Mechanics at the University of Minnesota. This article is based on the Theodore von Kármán Prize lecture that he gave in Chicago at the 2014 Annual Meeting.

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[*] Each of the 3×3 positive-definite, symmetric matrices $\mathbf{U}_1, \ldots \mathbf{U}_n$ has the same set of eigenvalues.

Taxi Sharing

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Given a shareability network, the problem of optimally matching taxi rides becomes equivalent to the well-known maximum matching problem on graphs, for which efficient solutions exist; this problem formulation allows us to compute the optimal matching of trips across the entire data set of more than 150 million trips. The shareability parameter k has a major impact on computational complexity: When $k > 2$, the shareability network becomes a hypergraph, and the problem of computing a maximum matching on the network becomes NP-hard.

The results of our study are extremely encouraging from the sharing economy viewpoint: With $k = 2$ and a passenger delay of at most 5 minutes, more than 95% of taxi rides in New York can be shared, resulting in a 30% reduction in the total travel time needed to accommodate all taxi requests and, as a consequence, a corresponding decrease in emissions.

The study reported in [5] is only a starting point in the quest for a deeper understanding of the benefits provided by the shared economy of mobility resources, and shareability networks can prove a valuable tool in this endeavor: The main idea, the translation of sharing into graph problems, might prove useful in analyzing other sharing scenarios in tomorrow's urban mobility landscape.

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Paolo Santi, who currently leads the MIT/Fraunhofer Ambient Mobility initiative at the Senseable City Lab of the Massachusetts Institute of Technology, is a senior researcher at the Istituto di Informatica e Telematica del CNR, Pisa, Italy.


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Bill Davidon: Back Stories

The late William Davidon, longtime professor of mathematics and physics at Haverford College, may be known to readers of *SIAM News* for his part in the DFP (Davidon–Fletcher–Powell) family of quasi-Newton methods. Those familiar with the *SIAM Journal on Optimization* might remember him as the author of the first paper—actually a reprint of his never-before-published 1959 Argonne report—in the first (February 1991) issue of the journal.

SIOPT founding editor John Dennis, explaining the choice of Davidon's then 30-year-old paper to launch the new journal, pointed out to *SIAM News* that "the line of research begun by Davidon dominated research in nonlinear programming for more than 20 years."

The story of the unhurried, indirect path to publication of the paper was recounted to *SIAM News* (July 1990) by the relaxed and unassuming Davidon, along with some remarkable side stories. The focus of those stories was Davidon's anti-Vietnam war activities, which included his spearheading of a 1971 break-in and burglary of the FBI office in Media, Pennsylvania. What motivated the burglary was Davidon's hope of obtaining and making public documents that would reveal attempts of the FBI to suppress dissent through surveillance and harassment of protesters. Downplayed by Davidon in the interview with *SIAM News*, the break-in and ensuing events are the subject of the recently published *Burglary: The Discovery of J. Edgar Hoover's Secret FBI* (by Betty Medsger, Vintage, 596 pages, $16.95, paper).

A review of the book in *The New York Review of Books* (October 23, 2014) concludes that the disclosure of the stolen material (originally in *The Washington Post*, where Medsger was a reporter), "helped to put a stop to many great abuses." In the end, the group did "a public service."

Experimental Math

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few areas that are particularly amenable to computational exploration—among them finite group theory, combinatorics and graph theory, number theory, evaluation of series and integrals. How can we expand the scope of questions that have been examined with these methodologies, not just to other areas of mathematics but to other fields as well?

All this also raises the question of how such work can be paid for. Unlike the case in the "hard sciences," the majority of published mathematical research (pure and applied) is completed without direct research funding, by academic mathematicians or others as they have time alongside their teaching or other formal duties. But some of the work described here, particularly that involving substantial software development and maintenance, cannot be done so informally. Nor does a royalty model work, as it has for traditional publications—the development costs are too great and the academic rewards too small.

It is clear that researchers in experimental mathematics need to work more vigorously with government funding agencies to find ways to provide this funding. Perhaps this may be done more easily if projects can be pursued in collaboration with researchers in other disciplines, particularly in fields such as computer science that have typically been somewhat more generously funded.

David H. Bailey is a retired senior scientist at the Lawrence Berkeley National Laboratory and a Research Fellow at the University of California, Davis. Jonathan M. Borwein is Laureate Professor in the School of Mathematical and Physical Sciences at the University of Newcastle and director of the university's Priority Research Centre in Computer Assisted Research Mathematics and its Applications (CARMA).

Future Cities

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In the spirit of micro-level digital interactions, one of us has initiated a LinkedIn group on MSSC: Mathematical Sciences for Smart Cities. Interested readers are encouraged to join us.

Acknowledgments

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Peter Grindrod is a professor at the Mathematical Institute of the University of Oxford. Desmond J. Higham is a professor in the Department of Mathematics and Statistics at the University of Strathclyde. Robert S. MacKay is a professor in the Mathematics Institute and Centre for Complexity Science at the University of Warwick, UK.

Finite Elements

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$= 3 \times 10 = 30$. That dimension appears as the "atomic number" of the element at the upper right of the element box in Figure 1. It is a special case of the general formula

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

This formula, and similar ones for the other three families, appear in the gray family-description boxes near the top of the poster. Tables of values of the dimensions, for n up to 4 and r up to 7, appear at the bottom of the poster. In addition to the dimension and nomenclature for the elements, the element boxes display information about the degrees of freedom of the elements, and how the elements can be used in computer code.

Degrees of freedom. In addition to the shape functions of a finite element, we must specify its degrees of freedom (DOFs). These are a unisolvent set of functionals on the shape functions, with each functional associated to a face of some dimension (e.g., a vertex, edge, 2-face). The DOFs specify how the polynomial pieces are pasted together in a manner that can be efficiently implemented. In constructing the global finite element space from the polynomial pieces (the shape functions), the associated DOFs are constrained to take the same values whenever two elements share a common face. In this way the choice of DOFs imposes a certain degree of continuity on the finite element space.

As shown in Figure 1, for the $\mathcal{P}_2\Lambda^1$ element, 3 DOFs are associated to each of the 6 edges of the tetrahedron, and another 3 to each of its 4 faces, but none are associated to the vertices or to the tetrahedron itself. Because the full set of DOFs is a basis for the dual space of the space of shape functions, their number must equal the dimension of the shape function space, which is reflected in the calculation $6 \times 3 + 4 \times 3 = 30$. The equation

$$6\times\underbrace{\mathcal{P}_2^-\Lambda^0(\Delta_1)}_{3}+4\times\underbrace{\mathcal{P}_1^-\Lambda^1(\Delta_2)}_{3}=30 \quad (1)$$

displayed in the element box for the $\mathcal{P}_2\Lambda^1$ element includes this information, along with additional information about the DOFs.

For all of the elements in the four families, the DOFs are weighted moments. More precisely, the DOFs associated to a face f of dimension d are functionals acting on a k-form u as

$$u \mapsto \int_f (\mathrm{tr}_f u) \wedge q \qquad (2)$$

for an appropriate set of weight functions q. The weight functions are differential ($d - k$)-forms on f; that is, $q \in \Lambda^{d-k}(f)$. Then, because the trace $\mathrm{tr}_f u$ of a differential k-form u belongs to $\Lambda^k(f)$, the integrand $(\mathrm{tr}_f u) \wedge q \in \Lambda^{k+(d-k)}(f) = \Lambda^d(f)$, and so the integral makes sense. Notice that no DOFs are associated to faces of dimension $d < k$.

The choice of weighting functions is described on the poster for each of the four families. For example, the DOFs for the element $\mathcal{P}_r\Lambda^k(\Delta_n)$ are given by (2) with q belonging to the space $\mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$ (or more properly to a basis of that space). When $r + k - d < 1$, this space is to be interpreted as vanishing, and so there are DOFs only on the faces of dimension d with $k \le d \le r + k - 1$. Returning to our example with $k = 1$, $r = 2$, we see that there are indeed DOFs only on the faces of dimensions 1 and 2 (edges and triangles). On an edge f, the weights are given by the 3-dimensional space $\mathcal{P}_2^-\Lambda^0(f)$, while on a triangular face, the weights come from $\mathcal{P}_1^-\Lambda^1(f)$, which is again 3-dimensional. This is all captured in (1), which appears in the element box. Interestingly, the weight functions for the DOFs used to specify an element in the $\mathcal{P}$ family come from the shape function spaces of the $\mathcal{P}^-$ family, and vice versa.

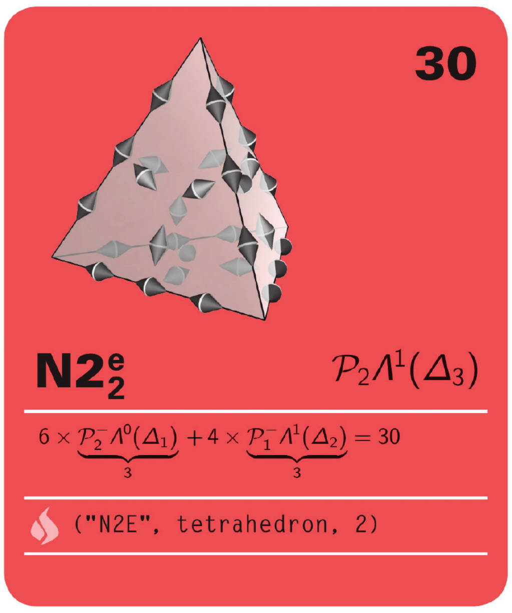

Figure 1. *An element box from the periodic table. This is the Nédélec second kind edge element of degree 2 or, in more mathematical nomenclature,* $\mathcal{P}_2\Lambda^1(\Delta_3)$.

Using the elements in computer code. The final component of the element box in Figure 1 is a snippet of computer code: `("N2E", tetrahedron, 2)`. This can be used to solve PDEs with the $\mathcal{P}_r\Lambda^k(\Delta_n)$ element in the FEniCS finite element software environment [8, 9, 10]. The element is instantiated there by calling `element = FiniteElement("N2E", tetrahedron, 2)`, which translates the element symbol $\mathbf{N2^e_2}$ into FEniCS syntax. Alternatively, FEniCS allows a syntax directly from the FEEC notation: `element = FiniteElement("P", tetrahedron, 2, 1)`.

Finite element spaces and their continuity. For each finite element in the table (choice of family, n, k, and r) and any n-dimensional mesh of simplices (for $\mathcal{P}^-$ or $\mathcal{P}$) or cubes (for $\mathcal{Q}^-$ or $\mathcal{S}$), we obtain a finite element space. This is a space of piecewise polynomial differential k-forms—that is, a space of scalar functions for $k = 0$ or n, a space of vector fields for $k = 1$ or $n - 1$.

The DOFs ensure that when two elements share a face, the traces of the corresponding shape functions agree. For 1-forms, for example, this means that the tangential components are continuous across faces, while for 2-forms in 3D it means the normal components are. This is reflected in the choice of symbol used to signify the DOFs in the element diagram. Continuity of the traces is exactly what is required to ensure that the finite element k-forms belong to the domain of the kth exterior derivative. For 0-forms, this means that the function is square integrable together with its gradient (which is the exterior derivative for 0-forms). Thus, the $k = 0$ spaces are spaces of H^1 finite elements, used, for example, to solve scalar second-order elliptic PDE problems, like the Poisson equation. These finite elements are continuous from element to element. By contrast, for $k = n$, no DOFs are specified on any faces of dimension $< n$, so no interelement continuity is imposed. These are L^2 finite elements, which are the basis of the discontinuous Galerkin methods. The case $k = n - 1$ gives $H(\mathrm{div})$ finite elements, piecewise polynomial vector fields whose normal component is continuous across element faces. These elements are very important in modern finite element methods, being used, for example, to solve the Darcy flow equations. In three dimensions, there remains the case of $k = 1$, which gives $H(\mathrm{curl})$ finite elements, crucial in electromagnetics.

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Finite Elements

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Differential Forms and Exterior Calculus

The organization and understanding of the periodic table depend heavily on differential forms and the primary operations on them, briefly discussed below.

Differential forms and exterior calculus unify numerous concepts of multivariable calculus in a fashion that applies to arbitrary manifolds. In exterior calculus, the fundamental object is a *differential k-form,* where the integer k ranges from 0 to the dimension n of the domain or manifold. 0-forms and 3-forms capture two different roles of scalar functions in trivariate calculus, while 1-forms and 2-forms give two viewpoints on vector fields. For example, 1-forms are the integrands of line integrals, and 2-forms are fluxes, which can be integrated over surfaces. Although Élie Cartan had not yet developed exterior calculus, Maxwell emphasized this distinction, writing that "physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area." Turning to scalar quantities, 0-forms are point functions whose gradients are 1-forms, while 3-forms are densities that can be integrated over spatial regions. In view of Stokes' theorem, it is not surprising that the curl of a 1-form results in a 2-form, while Green's theorem implies that the divergence of a 2-form gives a 3-form. All the relevant integrals (point evaluation, line integral, surface integral, and volume integral) are unified in the exterior calculus, and all three basic differential operators (grad, curl, and div) are subsumed in the exterior derivative d. Similarly, the various scalar and vector products of three-dimensional geometry are different cases of the wedge product of a k-form v and a j-form w, resulting in a $(j+k)$-form $v \wedge w$.

More precisely, a differential k-form is simply a function v that assigns to each point x of a manifold Ω an *algebraic* k-form on the tangent space $T_x\Omega$; that is, an alternating k-linear map $v_x : T_x\Omega \times \cdot\cdot\cdot \times T_x\Omega \to \mathbb{R}$. When Ω is a domain in $\mathbb{R}^n$, v is a function of $(k+1)$ variables. The first is the point x belonging to Ω, and the remaining ones are vectors belonging to $\mathbb{R}^n$. As a function of the final k variables, v is required to be linear and alternating, while as a function of x it is required only to possess some desired degree of smoothness (C^∞, C^0, L^p, . . .). In the special case $k = 0$, differential 0-forms are just real-valued functions on Ω. The space of differential k-forms on Ω is denoted $\Lambda^k(\Omega)$ (typically with C^∞ smoothness understood).

When the domain Ω is a subset of $\mathbb{R}^n$, differential forms can be viewed concretely through their coordinate representation. Let $\mathrm{d}x^i : \mathbb{R}^n \to \mathbb{R}$ denote the linear functional taking a vector to its ith coordinate. Then an algebraic 1-form (i.e., a linear functional) on $\mathbb{R}^n$ can be expressed as $\sum_{i=1}^{n} v_i \mathrm{d}x^i$ for some coefficients $v_i \in \mathbb{R}$. Allowing the v_i to depend on x, we obtain a differential 1-form. A basis for algebraic k-forms with $k > 1$ is obtained by taking the alternating part of the tensor product of k of the $\mathrm{d}x^i$. These are denoted by $\mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$, and so a differential k-form can be uniquely expressed as

$$v = \sum_{\sigma} v_\sigma \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k},$$

where the sum is over increasing sequences $1 \le \sigma_1 < \cdot\cdot\cdot < \sigma_k \le n$ and the $\binom{n}{k}$ coefficients v_σ are real-valued functions.

The *exterior derivative* of a differential k-form v is the $(k+1)$-form

$$\mathrm{d}v = \sum_{\sigma} \sum_{j=1}^{n} \frac{\partial v_\sigma}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k},$$

while for a k-dimensional submanifold $\omega \subset \Omega$, the *integral*

$$\int_\omega v = \sum_{\sigma} \int_\omega v_\sigma \mathrm{d}x^{\sigma_1} \wedge \cdots \wedge \mathrm{d}x^{\sigma_k}$$

is a real number, defined up to a sign that is fixed by the choice of an orientation of ω.

Returning to the case of $\mathbb{R}^3$, a scalar function v can be viewed as either a 0-form (itself) or a 3-form, $v\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. A vector field (v_1, v_2, v_3) corresponds to either the 1-form $v_1\,\mathrm{d}x^1 + v_2\,\mathrm{d}x^2 + v_3\,\mathrm{d}x^3$ or the 2-form $v_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 - v^2\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + v^3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$. The wedge product of a 0-form and a k-form is the scalar product; the wedge product of two 1-forms is the cross product; and the wedge product of a 1- and a 2-form is the dot product (in other cases the result is 0). The exterior derivative on k-forms is the gradient, curl, divergence, and zero, respectively, for $k = 0, . . ., 3$, while the integral corresponds to point evaluation for 0-forms, the (tangential) line integral for 1-forms, the (normal) surface integral for 2-forms, and the volume integral for 3-forms.

The online version of this article (posted at sinews@siam.org) contains additional material about each of the four finite element families. The references in the print version that are not cited in the text appear in the web version.

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Douglas Arnold is the McKnight Presidential Professor of Mathematics in the School of Mathematics at the University of Minnesota. Anders Logg is a professor of computational mathematics in the Department of Mathematical Sciences at Chalmers University of Technology and the University of Gothenburg.

Professional Opportunities

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Students (and others) in search of information about careers in the mathematical sciences can click on "Careers and Jobs" at the SIAM website (www.siam.org) or proceed directly to

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Institute for Advanced Study

School of Mathematics

The School of Mathematics at the Institute for Advanced Study, in Princeton, New Jersey, will have a limited number of memberships with financial support for research during the 2015–2016 academic year. School term dates for 2015–2016 are: Term I, Monday, September 21 to Friday, December 18, 2015; Term II, Monday, January 11 to Friday, April 8, 2016. Applicants should note that the school's Term II begins and ends one week later than the rest of the institute's. The school frequently sponsors special programs; however, these programs comprise no more than one-third of the membership so that a wide range of mathematics can be supported each year.

During the 2015–2016 academic year, the school will have a special program on geometric structures on 3-manifolds. Ian Agol, of the University of California, Berkeley, will be the Distinguished Visiting Professor. Thurston proposed classification of geometric structures on n-manifolds. While the spectacular geometrization theorem classified the geometric structures on 3-manifolds with compact isotropy group, i.e., locally homogeneous Riemannian metrics, there is a cornucopia of other fascinating structures, such as contact structures, foliations, conformally flat metrics, and locally homogeneous (pseudo-) Riemannian metrics. The goal of this program is to investigate these other geometric structures on 3-manifolds and to discover connections between them. Additionally, it is important to forge connections between geometric structures on 3-manifolds and other geometric constructs, such as gauge theory, PD (3) groups, minimal surfaces, cube complexes, geometric structures on bundles over 3-manifolds, and strengthened structures, such as taut foliations, tight contact structures, pA flows, convex projective structures, and quasi-geodesic foliations. Many of these do not even have a conjectural classification (in terms of topological restrictions and moduli), and specific examples are still being constructed.

Applicants must give evidence of ability in research comparable with at least that expected for a PhD degree but can otherwise be at any career stage. Successful candidates will be free to devote themselves full time to research. About half of the school's members will be postdoctoral researchers within five years of receipt of a PhD. The school also expects to offer some two-year postdoctoral positions.

Applications are invited for up to eight von Neumann Fellowships that are available each academic year; to be eligible for a von Neumann Fellowship, applicants should be at least five, but no more than 15, years after receipt of a PhD.

Applicants can also apply for Veblen Research Instructorships, which are three-year positions that were established in partnership with the Department of Mathematics at Princeton University. These instructorships are offered each year to candidates in pure and applied mathematics who have received a PhD within the last three years. Usually, Veblen research instructors spend their first and third years at Princeton University; these years will carry regular teaching responsibilities. The second year is spent at the institute and dedicated to independent research of the instructor's choice. Applicants interested in a Veblen instructorship position can apply directly at the IAS website (https://applications.ias.edu) or through MathJobs, https://www.mathjobs.org/jobs. Applicants applying through MathJobs must also complete an application form at https://applications.ias.edu; however, they do not need to submit a second set of reference letters. Applicants who have questions about the application procedure can e-mail applications@math.ias.edu.

Applications are also invited for two-year postdoctoral positions in computer science and discrete mathematics to be offered jointly with one of the following: Department of Computer Science at Princeton University, http://www.cs.princeton.edu; DIMACS at Rutgers, The State University of New Jersey, http://www.dimacs.rutgers.edu; or the Intractability Center, http://intractability.princeton.edu. For a joint appointment applicants must apply to the IAS, as well as to one of the listed departments or centers, indicating their interest in a joint appointment. The deadline for all applications is December 1, 2014.

The Institute for Advanced Study is committed to diversity and strongly encourages applications from women and minorities.

Dartmouth College

Department of Mathematics

New or recent PhD graduates with research interest in applied and computational mathematics may apply for instructorships in these areas for terms of two to three years. Successful candidates will teach three 10-week courses spread over three terms. Appointments are for 26 months, with a possible 12-month renewal. Positions offer a monthly salary of $5,202, which includes a two-month research stipend for instructors in residence during two of the three summer months; if an instructor is not in residence, the salary will be adjusted accordingly.

To initiate an application, applicants should go to http://www.mathjobs.org and find position ID IACM #6022. The application can also be accessed at http://www.math.dartmouth.edu/activities/recruiting/. General inquiries can be directed to Tracy Moloney, Administrator, Department of Mathematics, tfmoloney@math.dartmouth.edu. Applications completed by January 5, 2015, will be considered first.

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Dartmouth College

Department of Mathematics

The Department of Mathematics anticipates a senior opening with an initial appointment in the 2015–2016 academic year. The successful applicant will have a research profile with a concentration in computational or applied mathematics, will be appointed at the level of full professor, and is expected to have an overall record of achievement and leadership consonant with such an appointment.

Applicants should apply online at www.mathjobs.org (position ID: PACM #6023). Applications received by December 15, 2014, will receive first consideration. For more information about this position, please visit http://www.math.dartmouth.edu/activities/recruiting/.

Dartmouth is committed to diversity and encourages applications from women and minorities.

Dartmouth College

Department of Mathematics

John Wesley Young Research Instructorships are available for two to three years for new or recent PhD graduates whose research overlaps a department member's. Successful candidates will teach three 10-week courses spread over three terms. Appointments are for 26 months, with a possible 12-month renewal; the monthly salary is $5,202, including a two-month research stipend for instructors in residence during two of three summer months. If an instructor is not in residence, the salary will be adjusted accordingly.

To initiate an application go to http://www.mathjobs.org (position ID: JWY #6021). The application can also be accessed through a link at http://www.math.dartmouth.edu/activities/recruiting/. General inquiries can be directed to Tracy Moloney, Administrator, Department of Mathematics, tfmoloney@math.dartmouth.edu. Applications completed by January 5, 2015, will be considered first.

Dartmouth College is committed to diversity and strongly encourages applications from women and minorities.

Dartmouth College

Department of Mathematics

The Dartmouth College Department of Mathematics is pleased to announce a tenure-track opening for the academic year 2015–2016. There is a preference for a junior appointment, but appointment at higher rank, with tenure, is possible. The successful candidate will have a research profile with a concentration in applied or computational mathematics.

Applicants should apply online at www.mathjobs.org (position ID: APACM #6024). Applications received by December 15, 2014, will receive first consideration. For more information about this position, please visit http://www.math.dartmouth.edu/activities/recruiting/.

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National University of Singapore

Department of Mathematics

The Department of Mathematics at the National University of Singapore (NUS) invites applications for tenured, tenure-track, and visiting positions at all levels, beginning in August 2015.

NUS is a research intensive university that provides quality undergraduate and graduate education. The Department of Mathematics has about 65 faculty members and teaching staff whose expertise covers major areas of contemporary mathematical research.

The department seeks promising scholars and established mathematicians with outstanding track records in any field of pure and applied mathematics. The department, housed in a newly renovated building equipped with state-of-the-art facilities, offers an internationally competitive salary with start-up research grants, as well as an environment conducive to active research, with ample opportunities for career development. The teaching load for junior faculty is kept especially light.

The department is particularly interested in, but not restricted to, considering applicants specializing in any of the following areas:

- partial differential equations and applied analysis;
- computational science, imaging and data science;
- operations research and financial mathematics;
- probability; or
- combinatorics.

Application materials (as PDF files) and enquiries should be sent to the Search Committee via email: search@math.nus.edu.sg.

Please include the following supporting documentation in the application:

See **Opportunities** *on page 10*

Opportunities

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1. NUS Personal Data Consent for Job Applicants: http://www.nus.edu.sg/careers/potentialhires/applicationprocess/NUS-Personal-Data-Consent-for-Job-Applicants.pdf;
2. an American Mathematical Society Standard Cover Sheet;
3. a detailed CV, including publications list;
4. a statement (max. of three pages) of research accomplishments and plan;
5. a statement (max. of two pages) of teaching philosophy and methodology. Please attach an evaluation on teaching from faculty members or students of applicant's current institution, where applicable; and
6. at least three letters of recommendation, including one that indicates the applicant's effectiveness in and commitment to teaching. Reference letters should be sent directly to search@math.nus.edu.sg.

The review process will begin on October 15, 2014, and will continue until positions are filled.

For further information about the department, please visit http://www.math.nus.edu.sg.

Georgia Institute of Technology

School of Mathematics

The School of Mathematics at Georgia Tech is accepting applications for faculty positions at all ranks and in all areas of pure and applied mathematics and statistics.

Applications by highly qualified candidates, especially those from groups underrepresented in the mathematical sciences, are particularly encouraged. See www.math.gatech.edu/resources/employment for more details and application instructions.

California Institute of Technology

Department of Computing and Mathematical Sciences

The Department of Computing and Mathematical Sciences (CMS) at Caltech invites applications for a tenure-track faculty position. The department is a unique environment where innovative, interdisciplinary, and foundational research is conducted in a collegial atmosphere. The department seeks candidates who have demonstrated exceptional promise through novel research with strong potential connections to natural, information, and engineering sciences. Research areas of particular interest include applied mathematics and computational science as well as computing. A commitment to high-quality teaching and mentoring is expected.

The initial appointment at the assistant-professor level is for four years and is contingent upon the completion of a PhD in applied mathematics, computer science, or a related field. Exceptionally well-qualified applicants may also be considered at the full professor level.

To ensure the fullest consideration, applicants are encouraged to have all their application materials on file by December 28, 2014. For a list of documents required and full instructions on how to apply online, visit http://www.cms.caltech.edu/search. Questions about the application process may be directed to: search@cms.caltech.edu.

Caltech is an Equal Opportunity/Affirmative Action Employer. Women, minorities, veterans, and disabled persons are encouraged to apply.

Southern Methodist University

Department of Mathematics

Applications are invited for the Clements Chair of Mathematics (position no. 00050961) to begin in the fall semester of 2015. The department is searching for senior scholars with outstanding records of research in computational and applied mathematics as well as a strong commitment to teaching, including an established history of advising doctoral students. The department seeks candidates whose interests align with those of the department and who would contribute in a substantial way to the university's initiatives in high-performance computing and interdisciplinary research. In addition, the Clements Chair is expected to provide leadership in the further development of the department's graduate and undergraduate programs.

The Department of Mathematics offers graduate degrees in computational and applied mathematics and includes 16 tenured or tenure-track faculty researchers, all of whom work in application areas. Visit http://www.smu.edu/math/ for more information.

To apply send a letter of application with a curriculum vitae, a list of publications, research and teaching statements, and the names of three references to: The Faculty Search Committee, Department of Mathematics, Southern Methodist University, P.O. Box 750156, Dallas, Texas, 75275-0156. The Search Committee can also be contacted via email, phone, or fax: mathsearch@mail.smu.edu; phone: (214)768-2452; fax: (214)768-2355. A PhD in applied mathematics or a related field is required.

Applications received by December 1, 2014 will receive full consideration, but applications will continue to be accepted until the position is filled. Applicants will be notified when the search is concluded.

SMU, a private university with active graduate and undergraduate programs in the sciences and engineering, is situated in a quiet residential section of Dallas, Texas. The Dallas–Fort Worth Metroplex is America's fourth largest metropolitan area, and residents enjoy access to world-class cultural and entertainment activities.

SMU will not discriminate on the basis of race, color, religion, national origin, sex, age, disability, genetic information, or veteran status. SMU's commitment to equal opportunity includes nondiscrimination on the basis of sexual orientation and gender identity and expression. Hiring is contingent upon the satisfactory completion of a background check.

Brown University

Institute for Computational and Experimental Research in Mathematics (ICERM)

The Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University invites applications for its postdoctoral fellowship positions. ICERM's two postdoctoral institute fellowships are 9-month salaried positions (with the possibility of summer support), both commencing in September 2015. One will participate in the fall 2015 "Computational Aspects of the Langlands Program" semester program (http://icerm.brown.edu/sp-f15/) and remain as a researcher-in-residence during the spring 2016 semester. The other will begin as a researcher-in-residence during the fall 2015 semester and will participate in the spring 2016 "Dimension and Dynamics" semester program (http://icerm.brown.edu/programs/sp-s16/).

ICERM's eight postdoctoral fellowships are semester-long positions that come with stipends. Four begin in September 2015 during the "Computational Aspects of the Langlands Program" semester program. The other four start in February 2016 during the "Dimension and Dynamics" semester program.

All postdoctoral fellows are matched with faculty advisors.

Eligible applicants must have completed their PhD within three years of the start of the appointment. Applicants must submit an AMS Standard Cover Sheet, curriculum vitae (including publication list), cover letter, research statement, and three letters of recommendation via Mathjobs.org (http://www.mathjobs.org/jobs/ICERM). Applications will be accepted until all positions are filled.

Boston University

Department of Mathematics and Statistics

The Department of Mathematics and Statistics at Boston University invites applications for a tenure-track assistant professor in geometry and mathematical physics. A PhD is required, and salary will be commensurate with experience. The position will begin in July 2015. Strong commitment to research and teaching at the undergraduate and graduate levels is essential.

Please submit all materials to mathjobs.org. Alternatively send a cover letter, curriculum vitae, research statement, teaching statement, and at least four letters of recommendation, one of which addresses teaching, to: Geometry and Mathematical Physics Search, Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, MA 02215. The application deadline is December 15, 2014.

Boston University is an equal opportunity employer, and all qualified applicants will receive consideration for employment without regard to race, color, religion, sex, national origin, disability status, protected veteran status, or any other characteristic protected by law. The university is a VEVRAA Federal Contractor.

Boston University

Department of Mathematics and Statistics

The Department of Mathematics and Statistics at Boston University invites applications for a tenure-track assistant professor in probability, stochastic processes and statistics. A PhD is required, and salary will be commensurate with experience. The position will begin in July 2015. Strong commitment to research and teaching at the undergraduate and graduate levels is essential.

Please submit all materials to mathjobs.org. Alternatively send a cover letter, curriculum vitae, research statement, teaching statement, and at least four letters of recommendation, one of which addresses teaching, to: Probability, Stochastic Processes, and Statistics Search, Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, MA 02215. The application deadline is December 15, 2014.

Boston University is an equal opportunity employer, and all qualified applicants will receive consideration for employment without regard to race, color, religion, sex, national origin, disability status, protected veteran status, or any other characteristic protected by law. The university is a VEVRAA Federal Contractor.

Bates College

Department of Mathematics

The Bates College Department of Mathematics invites applications for a tenure-track position at the rank of assistant professor, beginning August 1, 2015. Applicants are particularly welcome in the field of computational/applied mathematics, with research focusing on areas such as scientific computation, bioinformatics, mathematical finance and economics, or others. Applicants should have a commitment to undergraduate education in a liberal arts college setting and should show promise of excellence and innovation in both teaching and scholarship. The teaching load is five courses per academic year, distributed across two 12-week semesters and one 5-week spring term.

Review of applications begins November 15, 2014, and will continue until the position is filled. Preference will be given to candidates who will have completed a PhD or equivalent degree in mathematics, applied mathematics, or other appropriate field by August 1, 2015.

Applicants should submit all requested materials in PDF format to apply.interfolio.com/26865. Only the documents requested in this ad will be considered in the review of applications. Applications should include the following:

- A cover letter that addresses the applicant's interest in working at a small, residential, liberal arts college;
- a CV;
- a teaching statement that includes a description of how the applicant's teaching can contribute to a learning community that values diversity and inclusion;
- a research statement that describes the applicant's work to a hiring committee drawn from a broad mathematical audience; and
- a graduate school transcript.

Applicants must also arrange for three letters of recommendation, at least one of which addresses

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Postdoctoral fellowships are available (up to 6) at the Statistical and Applied Mathematical Sciences Institute (SAMSI) for the two SAMSI Research Programs for 2015-2016 **Challenges in Computational Neuroscience (CCNS)** and **Statistics and Applied Mathematics in Forensic Science (Forensics)**. Appointments will begin in August 2015 and will typically be for two years, although they can also be arranged for one year. Appointments are made jointly between SAMSI and one of its partner universities, where teaching opportunities may be available. Extremely competitive salaries, travel stipend, and health insurance will be offered.

Criteria for selection of SAMSI Postdoctoral Fellows include demonstrated research ability in statistical and/or applied mathematical sciences, computational skills along with good verbal and written communication abilities, and finally, a strong interest in the SAMSI program areas. The deadline for full consideration is December 15, 2014, although later applications will be considered as resources permit.

In your cover letter, please specify which of the two SAMSI research programs you are applying to (CCNS or Forensics) and why you would be a good fit for SAMSI and that program.

To apply, go to mathjobs.org, SAMSIPD2015 Job #6133

CHAIR

Department of Mathematics

The University of Alabama seeks an outstanding individual at the rank of Associate or Full Professor for the position of Chair of the Department of Mathematics. The successful candidate must be nationally/internationally recognized, with an active research program that includes external funding, and with the ability to help shape a progressive thriving department within a university whose student enrollment has nearly doubled in the last ten years and whose trajectory is upwards. The applicant should possess proven leadership abilities, preferably with administrative experience, and have an understanding and enthusiasm for both the teaching and research missions. The area of expertise of the applicant is open, but should complement those of the existing faculty and future plans for growth in the Department.

The University of Alabama is the flagship campus of a three-campus system. The University is located in Tuscaloosa, a city of approximately 100,000. The UA Department of Mathematics has 28 tenured/tenure track faculty, 11 full-time instructors and 40 graduate students, with research programs in algebra, analysis, fluid dynamics, image processing, mathematics education, optimization, scientific computing, statistics, stochastic processes and topology. The department has an in-house Mathematics Ph.D. program and a joint Applied Mathematics Ph.D. program with the other two campuses. The UA Mathematics program places emphasis on quality education at the undergraduate and graduate levels.

Applicants should apply online at **https://facultyjobs.ua.edu/postings/35868**; attach a curriculum vita along with a letter of application and arrange for three letters of recommendation to be sent to **math@ua.edu**. Statements of administrative and leadership philosophy, research plans, and teaching philosophy and interests should also be included. Potential candidates may contact the chairperson of the search committee, Dr. Martyn Dixon, at **mdixon@ua.edu** if additional information is desired. Beginning October 15, 2014, applications and nominations will be reviewed on an ongoing basis and will continue to be accepted until the position is filled. The position is scheduled to start on August 16, 2015, or as negotiated.

For more information about the Department and the University visit our website at **http://www.math.ua.edu**.

The University of Alabama is an Equal Opportunity Employer/Affirmative Action employer and actively seeks diversity among its employees. Women and minority candidates are strongly encouraged to apply.

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THE UNIVERSITY OF ALABAMA

Opportunities

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the applicant's teaching experience or potential. These letters must be submitted through Interfolio in PDF format.

Please contact search committee chair Pallavi Jayawant (pjayawan@bates.edu) for more information. Do not send applications to Professor Jayawant; see application instructions above.

An equitable, inclusive and diverse campus and curriculum are critical to the educational mission of Bates College. Therefore, the college and the Department of Mathematics are committed to enhancing equity, inclusion, and diversity, including teaching students from all backgrounds. Applicants who can contribute to this goal are encouraged to apply, and the search committee expects applicants to identify their strengths and experiences in this area.

Bates is an Equal Opportunity/Affirmative Action employer. Because the college recognizes that employment decisions often involve two careers, Bates welcomes applications for shared positions. Employment is contingent on successful completion of a background check. For more information about the college, please visit www.bates.edu.

University of Colorado Denver

Department of Mathematical and Statistical Sciences

The Department of Mathematical and Statistical Sciences at the University of Colorado Denver invites applications for a tenure-track assistant professor position in numerical methods and scientific computing that begins August 2015. The university seeks candidates with excellent research potential and a strong commitment to quality teaching.

The application review begins November 15, 2014. For more information, see the full posting at www.jobsatcu.com (job posting F01791) or contact julien.langou@ucdenver.edu.

The University of Colorado Denver is committed to diversity and equality in education and employment.

Sandia National Laboratories

Computing Research Center and Computer Sciences and Information Systems Center

The Computing Research Center and the Computer Sciences and Information Systems Center at Sandia National Laboratories invite outstanding candidates to apply for the 2015 John von Neumann Postdoctoral Research Fellowship in computational science. This prestigious postdoctoral fellowship is supported by the Applied Mathematics Research Program in the U.S. Department of Energy's Office of Advanced Scientific Computing Research.

Sandia is one of the country's largest research facilities, employing nearly 8,700 people at major facilities in Albuquerque, New Mexico and Livermore, California. Sandia maintains research programs in a variety of areas such as computational and discrete mathematics, computational physics and engineering, and systems software and tools. Sandia is a world leader in large-scale parallel computer systems, algorithms, software, and applications, and provides a collaborative and highly multidisciplinary environment for solving computational problems at extreme scales. Sandia has a state-of-the-art parallel-computing environment consisting of advanced architectures, like the Cielo petascale machine, and numerous large-scale clusters and visualization servers, including the 264-TFlop Red Sky cluster and 392-TFlop Chama Cluster.

The fellowship provides an exceptional opportunity for innovative research in computational mathematics and scientific computing on advanced computing architectures with application to a broad range of science and engineering problems of national importance. Applicants must have or soon receive a PhD in applied/computational mathematics or related computational science and engineering disciplines. Applicants must have less than three years of postdoctoral experience. This appointment is for one year, with a possible renewal for a second year, and includes a highly competitive salary, moving expenses, and a generous professional travel allowance. For more details about the John von Neumann Fellowship, visit our website at www.cs.sandia.gov/VN_Web_Page.

To apply for the John von Neumann Fellowship, applicants should complete the following two steps:

(1) Submit a single PDF file containing a cover letter, CV, and research statement online at www.sandia.gov/careers, Job ID 647304. If an applicant does not receive information regarding the timeline for phone interviews within two weeks after submitting an application, please contact Denis Ridzal at dridzal@sandia.gov.

(2) Have three letters of recommendation sent to Denis Ridzal at dridzal@sandia.gov. Please ask references to use "2015 VN Fellowship" as the subject line.

Applications will be reviewed upon receipt. Complete applications received by December 1, 2014, will receive full consideration; the position will remain open until filled.

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College of Engineering: Open Rank Faculty

Department of Aerospace Engineering

College of Engineering

University of Illinois at Urbana-Champaign

The Department of Aerospace Engineering at the University of Illinois at Urbana-Champaign is seeking highly qualified candidates for multiple faculty positions with emphasis on the areas of space systems/propulsion, autonomous aerospace systems, multi-disciplinary design optimization, aeroelasticity, and aerospace materials and structures. Particular emphasis will be placed on qualified candidates who work in emerging areas of aerospace engineering and whose scholarly activities have high impact.

Please visit http://jobs.illinois.edu to view the complete position announcement and application instructions. Full consideration will be given to applications received by November 3, 2014. Applications received after that date will be considered until the positions are filled.

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ASSISTANT PROFESSOR

Scientific Computing Big Data

The Department of Mathematics at the University of Alabama invites applications for a tenure-track position at the assistant professor level in the general area of high-performance computing in data analysis beginning August 16, 2015. Candidates with interests in numerical linear algebra in data mining, optimization, statistical learning or cyber-security are encouraged to apply. Candidates must possess a doctorate in mathematics, statistics, or a closely related field. Applicants must apply online at **https://facultyjobs.ua.edu/postings/35996** and arrange for three letters of recommendation, one of which may address teaching, to be sent to **math@ua.edu**. The review process starts on December 1, 2014 and continues until the position is filled.

More information about the department and the university is available at **http://math.ua.edu**

The University of Alabama is an Equal Opportunity Employer/Affirmative Action employer and actively seeks diversity among its employees. Women and minority candidates are strongly encouraged to apply.

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City University of Hong Kong is a dynamic, fast-growing university that is pursuing excellence in research and professional education. As a publicly-funded institution, the University is committed to nurturing and developing students' talents and creating applicable knowledge to support social and economic advancement. The University has seven Colleges/Schools. As part of its pursuit of excellence, the University aims to recruit **outstanding scholars** from all over the world in various disciplines, including **business**, **creative media**, **energy**, **engineering**, **environment**, **humanities**, **law**, **science**, **social sciences**, **veterinary sciences** and other strategic growth areas.

Applications and nominations are invited for:

Chair Professor/Professor/Associate Professor/Assistant Professor Department of Mathematics *[Ref. A/152/49]*

Duties : Conduct research in areas of Applied Mathematics including Analysis and Applications, Mathematical Modelling (including biological/physical/financial problems), Scientific Computation and Numerical Analysis, and Probability and Statistics; teach undergraduate and postgraduate courses; supervise research students; and perform any other duties as assigned.

Requirements : A PhD in Mathematics/Applied Mathematics/Statistics with an excellent research record.

Salary and Conditions of Service

Remuneration package will be driven by market competitiveness and individual performance. Excellent fringe benefits include gratuity, leave, medical and dental schemes, and relocation assistance (where applicable). Initial appointment will be made on a fixed-term contract.

Information and Application

Further information on the posts and the University is available at http://www.cityu.edu.hk, or from the Human Resources Office, City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong [Email : hrojob@cityu.edu.hk/Fax : (852) 2788 1154 or (852) 3442 0311].

To apply, please submit an online application at http://jobs.cityu.edu.hk, and include a current curriculum vitae. Nominations can be sent directly to the Human Resources Office. **Applications and nominations will receive full consideration until the positions are filled.** Only shortlisted applicants will be contacted; and those shortlisted for the post of Assistant Professor will be requested to arrange for at least 3 reference reports sent directly by the referees to the Department, specifying the position applied for. The University's privacy policy is available on the homepage.

The University also offers a number of visiting positions for current graduate students, postdoctoral scholars, and for early-stage and established scholars, as described at http://www.cityu.edu.hk/provost/CityU_Visiting_Positions.htm.

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INSTITUTE FOR COMPUTATIONAL ENGINEERING & SCIENCES

The *Institute for Computational Engineering and Sciences (ICES)* at *The University of Texas at Austin* is searching for exceptional candidates with expertise in computational science and engineering to fill several Moncrief endowed faculty positions at the Associate Professor level and higher. These endowed positions will provide the resources and environment needed to tackle frontier problems in science and engineering via advanced modeling and simulation. This initiative builds on the world-leading programs at ICES in Computational Science, Engineering, and Mathematics (CSEM), which feature 16 research centers and groups as well as a graduate degree program in CSEM. Candidates are expected to have an exceptional record in interdisciplinary research and evidence of work involving applied mathematics and computational techniques targeting meaningful problems in engineering and science. For more information and application instructions, please visit: www.ices.utexas.edu/moncrief-endowed-positions-app/. This is a security sensitive position. The University of Texas at Austin is an Equal Employment Opportunity/Affirmative Action Employer.

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Los Alamos
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Los Alamos National Laboratory (LANL), a multidisciplinary research institution engaged in strategic science on behalf of national security, has a *single* open Center Leader position in the Center for Nonlinear Studies (CNLS). It will be filled either at the R&D Manager 4 or the R&D Scientist 5 level.

CENTER LEADER
(R&D MANAGER 4)
Job IRC34587

CENTER LEADER
(SCIENTIST 5)
Job IRC34586

The Center Leader provides scientific leadership and line management of the CNLS and plays an institutional and integrating role in collaboration with scientists throughout the Laboratory. The CNLS Center Leader is expected to develop and lead a program to target and create cooperative long-term research programs consistent with the Laboratory's strategic research objectives, to develop a strong working relationship with the CNLS External Advisory Committee, and to maintain effective working relationships throughout all levels of the Laboratory, government entities, academia and industry. The successful candidate will be expected to maintain an active research program while providing technical vision to nurture and support existing programs of others at the Center. The Center Leader should be energetic, results-oriented, a catalyst for change and an outstanding relationship builder. Line management responsibilities include accountability for quality research, management of financial and human resources, proactive support of Laboratory and Division safety, security, environment and diversity objectives, the communications/marketing strategy for the Center and collaboration with the Theoretical Division to help provide strategic direction for the organizations.

Position requires a Ph.D. degree in a scientific or engineering field relevant to the Center's activities and research or equivalent combination of education and experience. Demonstrated record of scientific accomplishment in one or more areas relevant to the Center as evidenced by an outstanding publication portfolio and/or a demonstrable national or international reputation is essential. Demonstrated experience in establishing and maintaining research collaborations, from the identification of new potential topics to forming teams, promoting proposals, and executing projects is also required.

Applicants may apply to both job postings at **careers.lanl.gov**

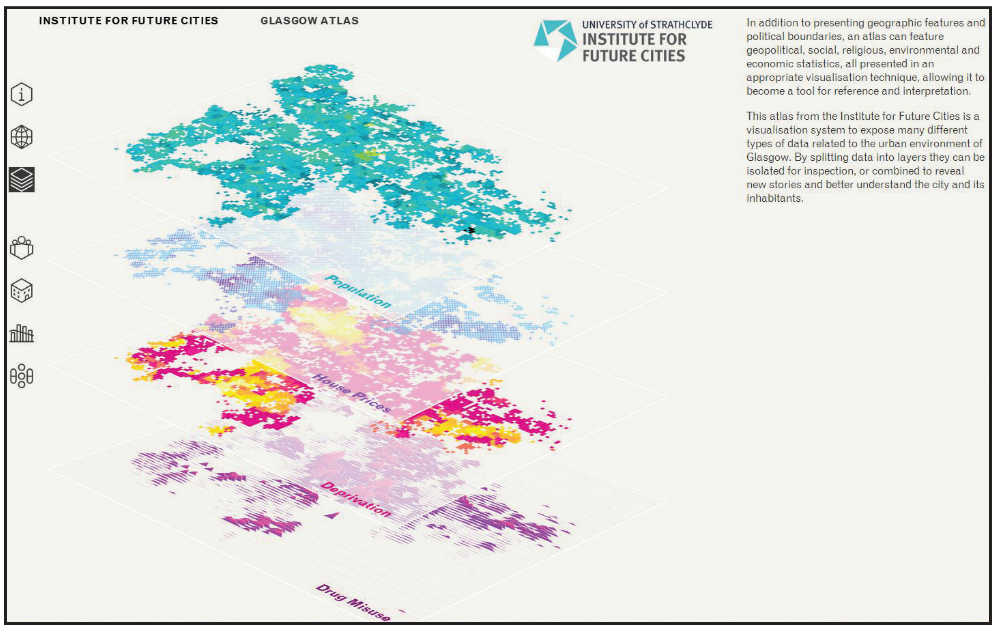
EOE

Opportunities at the Mathematics/Future Cities Interface

By Peter Grindrod,
Desmond J. Higham,
and Robert S. MacKay

More than half of the world's population lives in cities, a proportion that is estimated by the World Health Organization to reach 60% by 2030 and 70% by 2050. Thanks to the proliferation of smart devices and interconnected services, cities are gushing data, much of it related to human behavior. City life generates data streams around online social media, telecommunication, geolocation, crime, health, transport, air quality, energy, utilities, weather, CCTV, wi-fi usage, retail footfall, and satellite imaging. The powerful new concept of urban centres as "Living Labs" is inspiring novel research that could lead to improved well-being and economic growth.

We argue here that mathematicians can make an impact at the heart of this emerging interdisciplinary field, where hypotheses about human behavior must be quantified and tested against large-scale data sets, and where decisions and interventions should be based on quantitative, testable predictions. Moreover, the rapid growth of large-scale, disparate, multi-resolution data sets is driving new research challenges for applied and computational mathematicians, drawing on such hot topic areas as dynamic and multiplex networks (see Figures 1 and 2) [4, 9], multiscale modelling and simulation [2], uncertainty quantification [15], and sparse tensors [12, 13].



In addition to presenting geographic features and political boundaries, an atlas can feature geopolitical, social, religious, environmental and economic statistics, all presented in an appropriate visualisation technique, allowing it to become a tool for reference and interpretation.

This atlas from the Institute for Future Cities is a visualisation system to expose many different types of data related to the urban environment of Glasgow. By splitting data into layers they can be isolated for inspection, or combined to reveal new stories and better understand the city and its inhabitants.

Figure 1. *Multiplex visualization of population density, housing cost, deprivation, and drug-use levels across the city of Glasgow, Scotland. © LUSTlab/Institute for Future Cities, University of Strathclyde. Reproduced with permission.*

The Future Cities research arena we envisage is inherently interdisciplinary, encompassing the physical and social sciences, engineering, business, law, and, in particular, issues of privacy and ethics. At the risk of buzzword overload, we also note extensive overlap with other big-picture themes, including Data Science, Big Data, Complexity, Planet Earth, Digital Economy, the Internet of Things, and Computational Social Science.

Many urban centers around the world are becoming active in the Future Cities space, with strong support for these developments from governments and funding agencies. Focussing just on our home institutions, Glasgow City Council beat out 30 other cities to win a £24 million Future Cities Demonstrator competition, funded by the Technology Strategy Board, the innovation agency of the UK government; under this award, the Institute for Future Cities at the University of Strathclyde is developing a Digital Observatory that will allow public access to data generated in Glasgow and elsewhere. Future Cities is also one of the four strategic themes for Strathclyde's £89 million Technology and Innovation Centre, a hub for academic research and industrial collaboration. The University of Oxford's Engineering and Physical Sciences Research Council (EPSRC) Centre for Doctoral Training in New Industrially Focused Mathematical Modelling has a strong data analytics/technology component, and its Said Business School hosts the Institute for New Economic Thinking. The University of Warwick, which has designated Sustainable Cities as one of its Global Research Priorities, houses the Warwick Institute for the Science of Cities and offers an EPSRC Centre for Doctoral Training in Urban Science. The University of Warwick is also a partner in the Center for Urban Science and Progress (CUSP; http://cusp.nyu.edu/about/), a public/private research collaboration that uses New York City as a laboratory and classroom, and in its recently announced branch "CUSP London."

Further afield, Horizon 2020, the biggest European Union research and innovation programme to date, chose Societal Challenges as one of its three pillars, listing a €100 million call for research projects under the theme Smart Cities and Communities. In a draft strategic plan released in July 2014, EPSRC identified "designing and building future cities" as one of seven key challenges for the global economy.

A report commissioned by the UK Department for Business, Innovation and Skills [11] considered opportunities for UK industry in smart city technology across five urban market sectors—energy, water, transport, waste, and assisted living—estimating a global market of $408 billion by 2020.

Future Cities and the Math Sciences

We conclude with a brief look at recent developments and prospects in dynamical systems and in networks, with the focus mainly on our own research interests.

Macroscale observations have revealed scaling laws that relate city population size to such attributes as energy consumption, household income, and patent production, and important distinctions have been drawn between linear, sublinear, and superlinear growth [1, 14]. Explanatory, microscale models based on "hidden" laws must be consistent with such observations. Long-time dynamics and stability are key issues in the modelling of complex urban systems, as are sensitivities to parameter choices, including thresholds imposed by resource limitations [5]. In principle, good mathematical models can be used to map out ranges of possible behavior: An observed phenomenon might be constrained within a single domain of attraction (with others as yet unseen) and have a very low probability of breaking out; alternatively, it might reflect the trajectory of a chaotic process, where the qualitative macroscopic behaviour is predictable but the quantitative evolution of specific individuals is not (because of sensitive dependence on initial conditions and instability-driven disruptions).

In modelling terms, surgical extraction of the city from its surroundings may not be appropriate, and an open model, subject to a range of external influences, may be more realistic. Phenomena of interest might then be subject to persistent cycling or boiling, without ever approaching quiescence [17].

Digital interactions in an urban setting can naturally be represented as graphs, or networks, but the links between nodes in the system typically have an important time-dependent feature: Who just texted whom, who just logged on to which free wi-fi zone, who just reported a crime at which location? In a previous article in *SIAM News*, two of us discussed how a dynamic view of classical concepts in graph theory led to useful new algorithms [8]. But alongside the data-driven issue of extracting and summarizing information from network observations is the equally compelling challenge of deriving models that describe the underlying dynamics. Representing a network as a time-dependent matrix $A(t)$ whose (i,j) element quantifies the current level of interaction between nodes i and j, we can formalize concepts from the social sciences to derive suitable laws of motion (see sidebar).

In an urban context, where dynamic interactions take place on many levels between a range of parties, it is natural to think of dynamic models that operate across many layers, with the dynamics on one layer (e.g., the evolution of attitudes toward healthy lifestyle) coupled to the dynamics on another (e.g., the reach of a social media campaign). Moreover, with the advent of smartphones and GPS, we can now monitor geographic location across time and hence test models of urban movement [10].

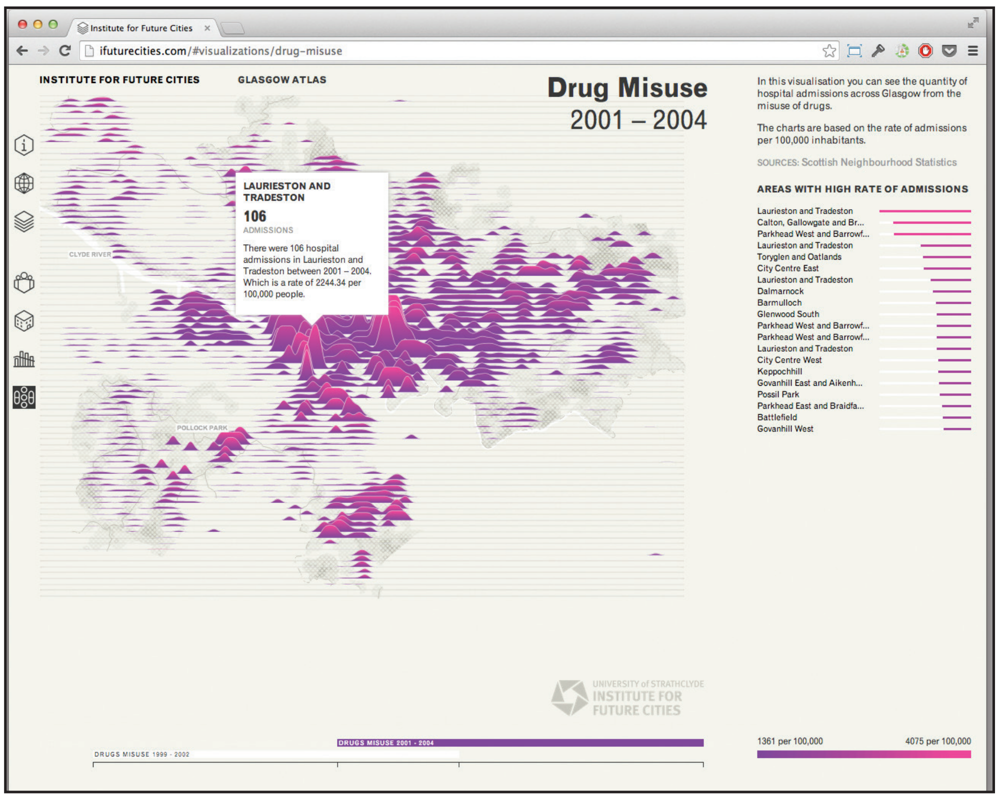
Social Balance

Traag, Van Dooren, and De Leenheer [16] looked at the concept of *social balance* (my friend's friend is my friend, my enemy's enemy is my friend, . . .) to derive matrix-valued ordinary differential equations of the form $\dot{A}(t) = A(t) \times A(t)$ and $\dot{A}(t) = A(t) \times A(t)^T$. Given such an $A(t)$, two of us [6] developed an accompanying ODE for the level of importance, or centrality, of the network nodes, showing that the matrix logarithm function arises naturally.

An alternative concept from the social sciences, triadic closure (the more friends I have in common with a person, the more likely I am to become the person's friend), was used in [7] to derive a stochastic birth and death model for link dynamics. There, results of mean-field analysis agreed with simulations showing that the network can self-organize into either of two very different long-term behaviors.

In the preamble to his recent book *The New Science of Cities* [3], Michael Batty of the Bartlett Centre for Advanced Spatial Analysis at University College London discusses three central principles that inform his "networks and flows" perspective of city science; all three resonate strongly with the standpoint of this article. Batty's first principle is that the relations between objects, not the intrinsic attributes of those objects, should condition our understanding, a viewpoint familiar to those who have been exposed to graph theory or category theory. Second, we should aim to measure, categorize, and look for universal scalings when we observe and compare city networks across space and time. Third, having gathered macro-level observations, we should seek to understand the micro-level principles that drive them—or, in the language of applied mathematics, we should aim for explanatory models, based on explicit modelling assumptions, with predictive power. Batty's book makes use of such concepts as agent-based modelling, flocking, graph theory, Markov chains, Markovian decision problems, optimization, and self-similarity/fractals, and hence is an excellent starting point for mathematicians wishing to enter the field.

See **Future Cities** *on page 7*



Drug Misuse
2001 – 2004

In this visualisation you can see the quantity of hospital admissions across Glasgow from the misuse of drugs.

The charts are based on the rate of admissions per 100,000 inhabitants.

SOURCES: Scottish Neighbourhood Statistics

AREAS WITH HIGH RATE OF ADMISSIONS

Laurieston and Tradeston
Calton, Gallowgate and Br...
Parkhead West and Barrowf...
Laurieston and Tradeston
Toryglen and Oatlands
City Centre East
Laurieston and Tradeston
Dalmarnock
Barmulloch
Glenwood South
Parkhead West and Barrowf...
Parkhead West and Barrowf...
Laurieston and Tradeston
City Centre West
Keppochhill
Govanhill East and Aikenh...
Possil Park
Parkhead East and Braidfa...
Battlefield
Govanhill West

LAURIESTON AND TRADESTON
106
ADMISSIONS
There were 106 hospital admissions in Laurieston and Tradeston between 2001 – 2004. Which is a rate of 2244.34 per 100,000 people.

Figure 2. *Reported levels of drug use across Glasgow, in a screen shot from the data streams of Figure 1. After discretisation based on, for example, city regions, combination of the levels in Figure 1 leads naturally to a three-dimensional tensor, with two dimensions representing spatial coordinates and the third dimension indexing the data sources. Time dependency in the data would add a fourth dimension. Extracting commonalities and differences, and summarising patterns, can be cast in terms of tensor factorisation—for example, generalising the matrix-level singular value decomposition. These four dimensions are not comparable—any results should be insensitive to the order in which we label the data streams, but for most purposes we should not reorder points in time or space. © LUSTlab/Institute for Future Cities, University of Strathclyde. Reproduced with permission.*