FINDING THE MAXIMUM MODULUS OF A POLYNOMIAL ON THE POLYDISK USING A GENERALIZATION OF STEČKINS LEMMA

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Abstract. This paper is a generalization of the work of J.J. Green in *Calculating the maximum modulus of a polynomial using Stečkin's Lemma.* This lemma is generalized to higher dimensions and is used in an algorithm to locate the absolute global max of a polynomial on the polydisk. How to apply this algorithm to the real sphere and the complex ball is also explained.

1. Introduction. The main goal of this paper is to show that the work of J.J. Green in [1] can be generalized to higher dimensions. Green developped an algorithm to locate the maximum modulus of a polynomial on the unit disk using a bound obtained with Stečkin's Lemma. Given an analytic polynomial p, we can define its maximum modulus on the unit disk

$$\|p\|_\infty := \sup\{|p(z)| : z \in \mathbb{C}, |z| \le 1\}$$

The maximum modulus principle[2] tells us that the maximum modulus is attained on the boundary of the domain of definition, thus in order to locate it, we can restrict ourselves at finding

$$\sup\{|p(e^{it})| : t \in [0, 2\pi]\}$$

The way Green finds this supremum is by subdividing $[0, 2\pi]$ into many small intervals, evaluating p at their middle point, and then looking if some bound is respected. If the bound is not respected for an interval, then this interval is discarded and the remaining intervals are subdivided into smaller ones, and so on. The bound he uses is a bound obtained by the following lemma[3]:

LEMMA 1.1 (Stečkin's Lemma). *Let $p(t) = \sum_{n=-d}^{d} c_n e^{int}$ be real trigonometric polynomial of degree d such that $\|p\|_\infty = p(0) = 1$, then for $|t| \le \frac{\pi}{d}$ we have*

$$p(t) \ge \cos(dt)$$

In order to apply this lemma to his problem, Green defined a new polynomial

$$q(t) = |p(e^{it})|^2 = p(e^{it})\overline{p(e^{it})}$$

and thus could derive a bound by substituting p for q in the lemma. What we will do in this paper is to generalize this process to higher dimensions, i.e. finding the maximum modulus of a multivariate polynomial on the polydisk, the real sphere and the complex ball, but first we need to explicit some definitions.

2. Definitions. Since we are working with multivariate polynomials in this paper, we will be using the multi-index notation. If $p(z) = \sum_{|n| \le d} c_n z^n$ is a multivariate polynomial, then

- $z = (z_1, \ldots, z_k)$

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- $n = (n_1, \dots, n_k)$
- $z^n = {z_1}^{n_1} \cdots {z_k}^{n_k}$
- $0 = (0, \dots, 0)$ and $1 = (1, \dots, 1)$
- $e^{it} = (e^{it_1}, \dots, e^{it_k})$
- $e^{int} = e^{i(n_1 t_1 + \cdots i n_k t_k)}$

The homogeneous degree of p is denoted

$$d = \max_n(|n|) \text{ where } |n| = |n_1| + \ldots + |n_k|$$

The polydisk is the k-dimensional generalization of the unit disk

$$\mathbb{D}^k = \{z \in \mathbb{C}^k : |z_j| \leq 1 \text{ for } j = 1 \ldots k\}$$

and its boundary, usually called the distinguished boundary, is

$$\begin{aligned}\mathbb{T}^k &= \{z \in \mathbb{C}^k : |z_j| = 1 \text{ for } j = 1 \ldots k\} \\ &= \{(e^{it_1}, \dots, e^{it_k}) \in \mathbb{C}^k : (t_1, \dots, t_k) \in [0, 2\pi]^k\}\end{aligned}$$

We will use the term maximum modulus of the polydisk for

$$\|p\|_\infty = \sup\{p(z) : z \in \mathbb{C}^k, |z_j| = 1 \text{ for } j = 1 \ldots k\}$$

3. Stečkin's Lemma generalization. This theorem is a very good estimate of the value of a trigonometric polynomial around a global maximum. Unfortunatly it has been proven only in the one-variable case. In order to find the maximum modulus of an analytic polynomial in k variables, we need to generalize this estimate. This requires a little bit of proof, so we start by proving a lemma that will be used in the proof.

LEMMA 3.1. *Let $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$, then there is $s \in \mathbb{R}, m \in \mathbb{Z}^k$ such that $|s| \leq \frac{\pi}{d \max_{1 \leq j \leq k} |m_j|}$ and $m_j s = \tilde{t_j}$ for $j = 1 \ldots k$, where $\tilde{t_j}$ is arbitrary close to t_j.*

Proof. Let $n \in \mathbb{N}$ be as big as wanted. Define $\tilde{t_j}^n$ to be t_j but with its decimal expansion truncated after the n^{th} decimal. Choose $s = 10^{-n}$ and $m_j = 10^n \tilde{t_j}^n$ then $m_j s = \tilde{t_j}^n$ for $j = 1, \dots, k$. We have

$$|s| \leq \frac{\pi}{d \max_{1 \leq j \leq k} |m_j|} \iff |m_j||s| \leq \frac{\pi}{d} \text{ for } j = 1, \dots, k.$$

But $|m_j||s| = |\tilde{t_j}^n| \leq |t_j| \leq \frac{\pi}{d}$ so it is true. Now, as n grows $\tilde{t_j}^n$ gets arbitrary close to t_j. ∎

Using this lemma we can now prove our generalized version of Stečkin's Lemma.

THEOREM 3.2 (Multivariate Stečkin's Lemma). *Let $p(t) = \sum_{|n| \leq d} c_n e^{int}$ be a real multivariate trigonometric polynomial of homogeneous degree d such that $p(0) = ||p||_\infty = 1$. Then for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$, we have*

$$p(t) \geq \cos(d \max_{1 \leq j \leq k} |t_j|)$$

Proof. Let $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$, then we have $m \in \mathbb{Z}^k$ and $s_0 \in \mathbb{R}$ such that

$$t = ms_0 \text{ and } |s_0| \leq \frac{\pi}{d \max_{1\leq j \leq k} |m_j|}$$

Define $q(s) = p(ms) = p(m_1 s, \ldots, m_k s) = \sum_{|n|\leq d} c_n e^{i(n_1 m_1 + \ldots + n_k m_k)s}$ which is a single variable real trigonometric polynomial of degree

$$\tilde{d} = \max_n |n_1 m_1 + \ldots + n_k m_k| \leq d \max_{1\leq j\leq k} |m_j|$$

Since $||q||_\infty \leq ||p||_\infty$ and $q(0) = p(0) = 1$ we conclude that $||q||_\infty = q(0) = 1$. Hence, by Stechkin's lemma we have

$$q(s) \geq \cos(\tilde{d}s) \text{ for } |s| \leq \frac{\pi}{\tilde{d}}$$

But since $|s_0| \leq \frac{\pi}{d \max_{1\leq j\leq k} |m_j|} \leq \frac{\pi}{\tilde{d}}$ we get

$$\begin{aligned} p(t) = p(ms_0) = q(s_0) &\geq \cos(\tilde{d}s_0) \\ &= \cos(\max_n |n_1 m_1 + \ldots + n_k m_k| s_0) \\ &= \cos(\max_n |n_1 m_1 s_0 + \ldots + n_k m_k s_0|) \\ &= \cos(\max_n |n_1 t_1 + \ldots + n_k t_k|) \\ &\geq \cos(d \max_{1\leq j\leq k} |t_j|) \end{aligned}$$

The last line is justified by the fact that

$$\pi \geq d \max_{1\leq j\leq k} |t_j| \geq \max_n |n_1 t_1 + \ldots + n_k t_k|$$

and $\cos(x)$ decreases on $[0, \pi]$. ☐

If we were using a different metric for the degree of the polynomial, we could derive an alternate version of the theorem as follows. Since the proof is almost identical, we will omit it.

THEOREM 3.3 (Multivariate Stečkin's Lemma 2). *Let $p(t) = \sum_{|n|\leq d} c_n e^{int}$ be a real multivariate trigonometric polynomial such that $|n_j| \leq d_j$ for $j = 1, \ldots, k$. If $p(0) = \|p\|_\infty = 1$, then for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$ such that $d_1|t_1| + \ldots + d_k|t_k| \leq \pi$ we have*

$$p(t) \geq \cos(d_1|t_1| + \ldots + d_k|t_k|)$$

4. Obtaining an optimal bound. Now that we have a generalized version of Stečkin's Lemma, we can derive a bound for analytic polynomials, so that we can apply it in our algorithm.

Suppose that we have $p(z) = \sum_{|n|\leq d} c_n z^n$ an analytic polynomial of homogeneous degree d such that $p(\xi) = \|p\|_\infty$, then we know by the maximum modulus principle that $\xi = e^{it_0} \in \mathbb{T}^k$. We define a new trigonometric polynomial

$$q(t) = 2\frac{|p(e^{i(t_0+t)}|^2}{\|p\|_\infty{}^2} - 1$$

It is important to note that the homegenous degree of q is d, because

$$\begin{aligned}
|p(e^{it})|^2 = p(e^{it}) * \overline{p(e^{it})} &= \sum_{|n|\le d} c_n e^{int} * \sum_{|n|\le d} \overline{c_n} e^{-int} \\
&= \sum_{\substack{|n_1|+\ldots+|n_k|\le d \\ |m_1|+\ldots+|m_k|\le d}} c_n\overline{c_m} e^{i(n_1t_1+\ldots+n_kt_k)} e^{i(-m_1t_1-\ldots-m_kt_k)} \\
&= \sum_{\substack{|n_1|+\ldots+|n_k|\le d \\ |m_1|+\ldots+|m_k|\le d}} c_n\overline{c_m} e^{i((n_1-m_1)t_1+\ldots+(n_k-m_k)t_k)}
\end{aligned}$$

and so the homogeneous degree of $|p(e^{it}|^2$ is $\max_{n,m}(|n_1-m_1|+\ldots+|n_k-m_k|)$ But since n_i and m_i are the power of z_i in the analytic polynomial $p(z)$, we conclude that $n_i \ge 0$ and $m_i \ge 0$ for $i = 1, 2, \ldots, k$ so really

$$\max_{n,m}(|n_1-m_1|+\ldots+|n_k-m_k|) \le \max_n(|n_1|+\ldots+|n_k|) = d$$

Now we have $|q(t)| \le 1$ and $q(0) = p(\xi) = 1$, hence q is a trigonometric polynomial of homegenous degree d such that $q(0) = ||q||_\infty = 1$ and we can apply the generalized version of Stečkin's Lemma to get for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$

$$\begin{aligned}
& q(t) \ge \cos(d \max_{1\le j\le k} |t_j|) \\
\iff\ & 2\frac{|p(e^{i(t_0+t)}|^2}{\|p\|_\infty{}^2} - 1 \ge \cos(d \max_{1\le j\le k} |t_j|) \\
\iff\ & \frac{|p(e^{i(t_0+t)}|}{\|p\|_\infty} \ge \sqrt{\frac{\cos(d\max_{1\le j\le k}|t_j|)+1}{2}}) \\
\iff\ & |p(e^{i(t_0+t)}| \ge \|p\|_\infty \cos\left(\frac{d\max_{1\le j\le k}|t_j|}{2}\right)
\end{aligned}$$

In the one-dimensional case, we get $|p(e^{i(t_0+t)}| \ge \|p\|_\infty \cos\left(\frac{dt}{2}\right)$ which is better than the bound $|p(e^{i(t_0+t)}| \ge \|p\|_\infty\sqrt{\cos(dt)}$ obtained by J.J. Green by defining $q(t) = |p(e^{it}|^2$.

5. The algorithm. Our algorithm works in the same spirit as Green's method. The idea is to split $[0, 2\pi]^k$ into many small cubes and evaluate p at their center. Then, using the bound we just found, we look if each of the small cube should be kept or rejected, and we subdivide further until enough precison is attained. Suppose that $\tilde{p}$ is the greatest measured absolute value of p on the polydisk. Let $H \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$ such that $|h_j| \le h$ for $j = 1\ldots k$. Let t be the center of the cube $C = [t-H, t+H]$. If the maximum modulus of p was to occur at e^{it_0} for $t_0 \in C$ then we would have $t - t_0 \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$ and hence

$$\begin{aligned}
|p(e^{it})| = |p(e^{i(t_0+(t-t_0))})| &\ge \|p\|_\infty \cos\left(\frac{d\max_{1\le j\le k}|t_j-t_{0j}|}{2}\right) \\
&\ge \|p\|_\infty \cos\left(\frac{d\max_{1\le j\le k}|h_j|}{2}\right) \ge \tilde{p}\cos\left(\frac{dh}{2}\right)
\end{aligned}$$

Now if for some center t, we have $|p(e^{it})| < \tilde{p}\cos\left(\frac{dh}{2}\right)$ then we know for sure that the maximum modulus cannot occur in that cube, so we can reject it. We also obtain the bounds $\tilde{p} \leq \|p\|_\infty \leq \tilde{p}\sec\left(\frac{dh}{2}\right)$ for the maximum modulus of p.

Using that rejection criteria, we can construct an algorithm to calculate the maximum modulus of a multivariate polynomial p on the polydisk that is described here.

Input: A multivariate polynomial p in k variables of homogeneous degree d, a desired precision of $\epsilon > 0$
Output: A close approximation of the maximum modulus of p on the polydisk.

$h \leftarrow \pi/d$
split $[0, 2\pi]^k$ equally into d^k small cubes of width $2h$
evaluate $|p(e^{it_i})|$ for each cube where t_i is the center of the cube
build a queue with all the cubes
$\tilde{p} \leftarrow \max\{|p(e^{it_i})| : i = 1, 2, \ldots, d^k\}$
repeat
  dequeue the cube in front of the queue
  $h_j \leftarrow$ width of the cube / 2
  $t_j \leftarrow$ center of the cube
  if $|p(e^{it_j})| \geq \tilde{p}\cos(\frac{dh_j}{2})$ **then**
    split this cube in 2^k smaller cubes and enqueue them
    if $|p(e^{it_j})| > \tilde{p}$ **then**
      $\tilde{p} \leftarrow |p(e^{it_j})|$
    end if
  else
    reject this cube
  end if
until $|\tilde{p}\sec(\frac{dh_j}{2}) - \tilde{p}| \leq \epsilon$
return $\frac{\tilde{p}+\tilde{p}\sec(\frac{dh_j}{2})}{2}$

6. Finding the maximum modulus on the real sphere. One could ask if it is possible to calculate the maximum modulus of a polynomial p on the k-dimensional real sphere like we have done for the polydisk, this is indeed possible using almost the same algorithm as for the polydisk. The k-dimensional real sphere is defined as

$$S^k = \{z \in \mathbb{R}^k : \sum_{j=1}^{k} |z_j|^2 = 1\}$$

One should note that we are not assuming that the maximum modulus of a real ball is taken on its boundary i.e. the sphere. Because of the fact that the real ball is a non-complex domain of definition, we can't apply the maximum modulus to it and so we only solve a simpler problem. We define the maximum modulus of a real sphere as

$$\hat{p} = \sup\{|p(z)| : z \in S^k\}$$

The next step is to define a mapping from the $(k-1)$-dimensional torus to the sphere which in our case will be defined as :

$$M : t \in [-\frac{\pi}{2}, \frac{\pi}{2}]^{k-2} \times [-\pi, \pi] \longmapsto z \in S^k$$
$$(t_1, ..., t_{k-1}) \longmapsto (w_1, \ldots, w_k)$$

where each of the w_j are defined by

$$w_1 = \sin(t_1)$$
$$w_j = \sin(t_j) * \prod_{i=1}^{j-1} \cos(t_i) \text{ for } j = 2 \ldots k-1$$
$$w_k = \prod_{i=1}^{k-1} \cos(t_i)$$

It can be shown by an easy but tedious induction that this mapping is one-to-one except at the poles of the sphere, where it is not injective. Using this map, we define a new trigonometric polynomial that we will use in order to apply Stečkin's Lemma. If we have an analytic polynomial p such that $\hat{p} = p(z_0) = p(M(t_0))$ then we define :

$$q(t) = 2\frac{|p(M(t_0 + t))|^2}{\hat{p}} - 1$$

which is a real trigonometric polynomial of a new degree $\tilde{d}$ such that $\hat{q} = q(0) = 1$. Now we apply exactly the same calculations as in section 4 to obtain for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]^k$ the bound

$$|p(M(t_0 + t))| \geq \|p\|_\infty \cos\left(\frac{\tilde{d}\max_{1\leq j\leq k}|t_j|}{2}\right)$$

Once we find what $\tilde{d}$ is, we can useexactly the same subdivision algorithm as in section 5 to locate and approximate the maximum modulus of p on the real sphere. We need a little bit of proof to show what the degree of q becomes.

LEMMA 6.1. *Let $p(z) = \sum_{|n|\leq d} c_n z^n$ be a multivariate polynomial of homogeneous degree d, then the polynomial $p(M(t))$ has an homogeneous degree*

$$\tilde{d} = \max_n(|n_k + n_{k-1}| + \sum_{i=1}^{k-1} |\sum_{j=i}^{k} n_j|)$$

Proof. We have

$$\begin{aligned}
p(M(t)) &= \sum_{|n|\leq d} c_n M(t)^n \\
&= \sum_{|n|\leq d} c_n \{w_1\}^{n_1}\{w_2\}^{n_2}\cdots\{w_k\}^{n_k} \\
&= \sum_{|n|\leq d} c_n \{\sin(t_1)\}^{n_1}\{\cos(t_1)\sin(t_2)\}^{n_2}\cdots\{\cos(t_1)\prod_{i=2}^{k-1}\cos(t_i)\}^{n_k}
\end{aligned}$$

We look at the power of each t_j individually. For the variable t_1, it appears in each of the parenthesis, so we conclude that the power of t_1 in $p(M(t))$ is $(n_1 + \ldots + n_k)$. The variable t_2 appears in every parenthesis but the first one so the power of t_2 is $(n_2 + \ldots + n_k)$.

And so on... except for the variables t_{k-1} and t_k which have the same power $(n_{k-1} + n_k)$. Now, by the definition of the homogeneous power, it is the sum of the individual absolute powers of t_j for each $j = 1 \ldots k-1$.

$$\begin{aligned}
&|n_k + \ldots + n_3 + n_2 + n_1| + \\
&|n_k + \ldots + n_3 + n_2| + \\
&|n_k + \ldots + n_3| + \\
&\ldots + \\
&|n_k + n_{k-1}| + \\
&|n_k + n_{k-1}| \\
&= |n_k + n_{k-1}| + \sum_{i=1}^{k-1} |\sum_{j=i}^{k} n_j|
\end{aligned}$$

The result follows from the definition of the homogenous degree. □

It is fairly clear that the homogenous degree of q is the same as $p(M(t))$. The value of $\tilde{d}$ could be found by a simple algorithm that looks at each of the homogeneous powers of the terms of p and then take the maximum.

For $t_1, t_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]^{k-2} \times [-\pi, \pi]$, the set $M([t_1, t_2])$ corresponds to an area on the surface of the sphere. The algorithm divides the sphere into many of these patches, evaluates p at their center and rejects or keep the patches depending on the estimate obtained by our bound. The problem with our mapping is that as we are getting closer to the poles of the sphere, the patches shrink dramatically, resulting in a much reduced efficiency at the poles. In the extreme case where

$$\hat{p} = p(\pm 1, 0, \ldots, 0) = p(M(\pm \frac{\pi}{2}, 0, \ldots, 0))$$

the algorithm has to keep a lot of regions around the pole where the maximum and that makes the algorithm become really slow. A solution that one could use to avoid this problem would be to make sure that $\hat{p}$ is not attained at a pole. One could start the algorithm and if for many iterations the only regions that are not excluded are mostly situated around the poles, then one could restart the algorithm with inducing a rotation to the sphere so that $\hat{p}$ is not situated at a pole anymore. This can be done by defining a new polynomial q as:

$$q(t) = 2\frac{|p(M(t_0 + t + R))|^2}{\hat{p}} - 1$$

where $R = (\frac{\pi}{4}, \frac{\pi}{2}, \ldots, \frac{\pi}{2})$ is the rotation applied to the sphere.

7. Applying the algorithm to the complex ball. We have done it for the polydisk and the real sphere, why not also do it for the complex ball. We have the

k-dimensional ball

$$B^k = \{z \in \mathbb{C}^k : \sum_{j=1}^{k} |z_j|^2 \leq 1\}$$

but with the maximum modulus principle, we can restrict ourselves at looking at its boundary, the complex sphere

$$\partial B^k = \{z \in \mathbb{C}^k : \sum_{j=1}^{k} |z_j|^2 = 1\}$$

and we define the maximum modulus of a polynomial p on the complex ball as

$$\hat{p} = \sup\{|p(z)| : z \in \partial B^k\}$$

Here again, we will map a torus to the complex sphere. The mapping we use is very similar to the real sphere, since a k-dimensional complex sphere can naturally be seen a $2k$-dimensional real sphere :

$$\begin{aligned} M_c : t \in [-\frac{\pi}{2}, \frac{\pi}{2}]^{2k-2} \times [-\pi, \pi] &\longmapsto z \in \partial B^k \\ (t_1, ..., t_{2k-1}) &\longmapsto (w_1 + w_2 i, \ldots, w_{2k-1} + w_{2k} i) \end{aligned}$$

where each of the w_j are defined by

$$\begin{aligned} w_1 &= \sin(t_1) \\ w_j &= \sin(t_j) * \prod_{i=1}^{j-1} \cos(t_i) \text{ for } j = 2 \ldots 2k-1 \\ w_{2k} &= \prod_{i=1}^{2k-1} \cos(t_i) \end{aligned}$$

In the same vein as for the real sphere, if we have a polynomial p such that $\hat{p} = p(z_0) = p(M_c(t_0))$ then we define a new polynomial

$$q(t) = 2\frac{|p(M_c(t_0 + t + R))|^2}{\hat{p}} - 1$$

where $R = 0$, or $R = (\frac{\pi}{4}, \frac{\pi}{2}, ..., \frac{\pi}{2})$ if it seems like $\hat{p}$ is taken at a pole. After finding the degree of q, we use again the calculations of section 4 and the algorithm of section 5 to locate the maximum modulus of p on the complex ball.

LEMMA 7.1. *Let $p(z) = \sum_{|n| \leq d} c_n z^n$ be a multivariate polynomial of homogeneous degree d, then the polynomial $p(M_c(t))$ has an homogeneous degree*

$$\tilde{d} = \max_n (|n_k| + 2\sum_{i=1}^{k-1} |\sum_{j=i}^{k} n_i|)$$

Proof. We have

$$p(M_c(t)) = \sum_{|n|\le d} c_n M_c(t)^n$$

$$= \sum_{|n|\le d} c_n \{w_1 + iw_2\}^{n_1} \cdots \{w_{2k-1} + iw_{2k}\}^{n_k}$$

$$= \sum_{|n|\le d} c_n \{\sin(t_1) + i\cos(t_1)\sin(t_2)\}^{n_1} \cdots \{\sin(t_{2k-1}) \prod_{i=1}^{2k-2} \cos(t_i) + i \prod_{i=1}^{2k-1} \cos(t_i)\}^{n_k}$$

We first look at the variable t_1. In each of the parenthesis, it appears in 2 terms of the same degree, so we need only consider one of them to find the degree of q in term of the variable t_1. By considering only one term in the last equation we obtain

$$p(M_c(t)) = \sum_{|n|\le d} c_n \{\sin(t_1) + \ldots\}^{n_1} \cdots \{\cos(t_1)(\cdots) + \ldots\}^{n_k}$$
$$= \sum_{|n|\le d} c_n \{\sin(t_1)^{n_1} \cdots \cos(t_1)^{n_k}(\cdots) + (\cdots)\}$$

Hence we conclude that the power of t_1 in q is $(n_1 + \ldots + n_k)$. Similarly for t_2 since it appears in every parenthesis. For t_3 and t_4, they appear in each of the parenthesis but the first one, so each of their power is $(n_2 + \ldots + n_k)$.

And so on... except for t_{2k-1} which appears only in the last parenthesis. So the power of t_{2k-1} is n_k. We now add the individual absolute powers of t_j for each $j = 1 \ldots k-1$:

$$\begin{aligned}
&2|n_k + \ldots + n_3 + n_2 + n_1| + \\
&2|n_k + \ldots + n_3 + n_2| + \\
&2|n_k + \ldots + n_3| + \\
&\ldots + \\
&2|n_k + n_{k-1}| + |n_k| \\
&= |n_k| + 2\sum_{i=1}^{k-1} |\sum_{j=i}^{k} n_i|
\end{aligned}$$

and the result follows from the definition of the homogenous degree. ☐

8. Notes on the complex ball. The complex sphere has a very specific structure that the real sphere doesn't have which induces on it different directions that one could exploit. There are 2 types of directions : the i×normal direction and the complex direction. We will note that going into the different directions may give different results.

Suppose that we have an analytic polynomial p such that $\|p\|_\infty = p(1, 0, \ldots, 0) = 1$ then we define the i×normal direction as:

$$(e^{it}, 0, \ldots, 0) : t \in [-\pi, \pi]$$

and the complex direction as :

$$(\cos(t), \sin(t), 0, \ldots, 0) : t \in [-\pi, \pi]$$

For the first one, we can use the original Stečkin's estimate to get for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]$

$$|p(e^{it}, 0, \ldots, 0)| \geq \cos\left(\frac{dt}{2}\right)$$

Along with this bound, we have the extremal polynomial $p(z) = \frac{1}{2}(1 + {z_1}^d)$. This polynomial is extremal in the i×normal direction because

$$\begin{aligned}
\left|p(e^{it}, 0, \ldots, 0)\right| &= \left|\frac{1}{2}(1 + e^{idt})\right| \\
&= \left|\frac{1}{2}(1 + \cos(dt) + i\sin(dt))\right| \\
&= \sqrt{\frac{1}{2}(1 + \cos(dt))} = \left|\cos\left(\frac{dt}{2}\right)\right| = \cos\left(\frac{dt}{2}\right)
\end{aligned}$$

for $t \in [-\frac{\pi}{d}, \frac{\pi}{d}]$, which is exactly the bound obtained by Stečkin's. By taking its power series, we can evaluate the decay of p as a function of d

$$\cos\left(\frac{dt}{2}\right) = 1 - \frac{d^2t^2}{8} + \ldots$$

and hence the decay of p in the direction of i×normal is in the order of d^2.

Now if we look at the complex direction, then we suspect that the polynomial $p(z) = ({z_1}^2 - {z_2}^2)^{\frac{d}{2}}$ is extremal, however, we can't show this fact here. If we work out the power series of p in the complex direction we find out that

$$\begin{aligned}
|p(\cos(t), \sin(t), 0, \ldots, 0) &= |(\cos(t)^2 - \sin(t)^2)^{\frac{d}{2}}| \\
&= \left|(\cos(t)^2 - (1 - \cos(t))^2)^{\frac{d}{2}}\right| \\
&= \left|(2\cos(t)^2 - 1)^{\frac{d}{2}}\right| \\
&= \left|\cos(2t)^{\frac{d}{2}}\right| \\
&= |1 - dt^2 + \ldots|
\end{aligned}$$

and we conclude that the decay of p in the complex direction is of the order of d. There is a difference in the order of the decays in the two directions, so there may be a better bound that one could obtain for the complex direction. However, we couldn't exploit this fact in our algorithm since we don't respect any of these directions while parametrizing the surface.

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