A connection between grad-div stabilized FE solutions and pointwise divergence-free FE solutions on general meshes

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Abstract

We prove, for Stokes, Oseen, and Boussinesq finite element discretizations on general meshes, that grad-div stabilized Taylor-Hood velocity solutions converge to the pointwise divergence-free solution (found with the iterated penalty method) at a rate of γ^{-1}, where γ is the grad-div parameter. However, pressure is only guaranteed to converge when $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition, where X_h is the finite element velocity space. For the Boussinesq equations, the temperature solution also converges at the rate γ^{-1}. We provide several numerical tests that verify our theory. This extends work in [6] which requires special macroelement structure in the mesh.

1 Introduction

It has recently been shown for several types of incompressible flow problems that on meshes with special structure that grad-div stabilized Taylor-Hood solutions converge to the Scott-Vogelius solution as the grad-div parameter approaches infinity [1, 7, 6]. Large grad-div stabilization parameters can be helpful in enforcing mass conservation in solutions, reducing error by reducing the effect of the pressure on the velocity error, and making the Schur complement easier to precondition and solve. However, a large parameter essentially enforces extra constraints on the velocity solution. Hence it is not clear if a negative effect can arise from a large parameter. By proving what the limit solution is, we can know the quality of a solution found with a large parameter.

The results of our paper extend this idea to general meshes where a Scott-Vogelius solution may not exist. In such cases, convergence will be to the pointwise divergence-free solution (which is found with the iterated penalty method). In this general case, the velocity solution still converges, but the pressure solution only converges when $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition, where X_h is the finite element velocity space; we consider X_h to be globally continuous piecewise polynomials. We consider the Stokes, Oseen, and Boussinesq systems, and for each case, we rigorously prove the convergence rates for velocity and pressure, and also temperature for Boussinesq systems. We numerically test our theory, and all computational results are consistent with our analysis.

This paper is arranged as follows: Section 2 provides notation and preliminaries to allow for a smooth presentation to follow. In Section 3, we study the Stokes equations. The main result is that the grad-div stabilized Taylor-Hood velocity solutions converge to the divergence-free solution with a rate of γ^{-1} as γ (the grad-div parameter) approaches ∞, and a modified pressure converges to the divergence-free pressure if $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition. In Section 4, we extend the results of Section 3 to the Oseen equations and obtain analogous results. Finally, in Section 5, we further extend the results to non-isothermic Stokes flows and again find analogous results.

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2 Mathematical preliminaries

We assume that the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is open and connected with Lipschitz boundary $\partial\Omega$. The $L^2(\Omega)$ norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We define the following function spaces

$$X := H_0^1(\Omega)^d,$$
$$Q := L_0^2(\Omega).$$

In X, it is well known that the Poincaré-Friedrich inequality holds: there exists a $C_{PF} = C_{PF}(\Omega)$ such that

$$\forall \phi \in X, \|\phi\| \leq C_{PF} \|\nabla \phi\|. \tag{2.1}$$

The dual of X is denoted by $H^{-1}(\Omega)$, with norm

$$\|\phi\|_{-1} := \sup_{v \in X} \frac{(\phi, v)}{\|\nabla v\|}.$$

We define the skew-symmetric trilinear operator $b^* : X \times X \times X \to \mathbb{R}$ by

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

From [5], we know that

$$|b^*(u, v, w)| \leq M \|\nabla u\| \|\nabla v\| \|\nabla w\|, \tag{2.2}$$

where M depends on the size of Ω.

Let $\mathcal{T}_h$ be a shape-regular, simplicial and conforming triangulation of Ω. Let $h_T = \text{diam}(T)$ denote the diameter of a simplex T and define $h := \max_{T \in \mathcal{T}_h}$. We denote by P_k the space of piecewise polynomials with respect to the triangulation $\mathcal{T}_h$ with degree not exceeding k, and by $P_k := [P_k]^d$ the analogous vector-valued polynomial space.

Throughout the paper, we consider finite dimensional spaces X_h, Q_h, defined by $X_h = X \cap \left(P_k(\mathcal{T}_h)^d\right)$ and $Q_h = Q \cap (P_{k-1}(\mathcal{T}_h))$, which are LBB-stable:

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(\nabla \cdot v_h, q_h)}{\|q_h\| \|\nabla v_h\|} \geq \beta > 0.$$

We define the space of discretely divergence-free functions

$$V_h := \{v_h \in X_h : \ (\nabla \cdot v_h, q_h) = 0 \ \forall q_h \in Q_h\},$$

and also the space of pointwise divergence-free functions,

$$V_h^0 := \{v_h \in X_h : \ \|\nabla \cdot v_h\| = 0\}.$$

Note that in general, and for most common element choices like Taylor-Hood, $V_h \neq V_h^0$. We also define the space orthogonal to V_h^0 in X_h in the X inner product:

$$R_h = (V_h^0)^\perp := \{r_h \in X_h, \ (\nabla r_h, \nabla v_h) = 0 \ \forall v_h \in V_h^0\}.$$

In R_h, we observe that the $L^2(\Omega)$ gradient norm and the divergence norm are equivalent [3], i.e. there exists C_R satisfying

$$\|\nabla v_h\| \leq C_R \|\nabla \cdot v_h\| \qquad \forall v_h \in R_h. \tag{2.3}$$

In the case that $V_h = V_h^0$, the constant C_R is independent of h [4], however in general it can depend inversely on h [3].

For our study of the Boussinesq system, we will also use a temperature space

$$S_h = \{s_h \in H^1(\Omega) \cap (P_k(\mathcal{T}_h)), s_h|_{\Gamma_{Dirichlet}} = 0\}$$

.

3 A connection for the Stokes equations

We first consider the Stokes equations, which are given by

$$\begin{aligned} -\nu\triangle u + \nabla p &= f, \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

where u represents velocity, p represents pressure, f is an external forcing, and ν is the kinematic viscosity. We will compare the grad-div stabilized Taylor-Hood scheme and the pointwise divergence-free scheme to determine a connection between the two methods. These two methods are given below:

Algorithm 3.1. *Given $\gamma \geq 0$, find $(w_h^\gamma, q_h^\gamma) \in (X_h, Q_h)$ satisfying*

$$\nu(\nabla w_h^\gamma, \nabla v_h) - (q_h^\gamma, \nabla \cdot v_h) + \gamma(\nabla \cdot w_h^\gamma, \nabla \cdot v_h) = (f, v_h) \ \forall v_h \in X_h, \tag{3.1a}$$

$$(\nabla \cdot w_h^\gamma, r_h) = 0 \qquad \forall r_h \in Q_h. \tag{3.1b}$$

Algorithm 3.2. *Find $u_h \in V_h^0$ satisfying*

$$\nu(\nabla u_h, \nabla v_h) = (f, v_h) \ \forall v_h \in V_h^0. \tag{3.2}$$

We will use the iterated penalty method to solve (3.2)*.*

$$\text{Step } 0: \ \nu(\nabla u_h^0, \nabla v_h) + (\alpha(\nabla \cdot u_h^0), \nabla \cdot v_h) = (f, v_h) \qquad \forall v_h \in X_h \tag{3.3a}$$

$$\text{Step } k: \ \nu(\nabla u_h^k, \nabla v_h) + (\alpha(\nabla \cdot u_h^k), \nabla \cdot v_h) = \nu(\nabla u_h^{k-1}, \nabla v_h) \ \forall v_h \in X_h \tag{3.3b}$$

If the iterated penalty method converges, which is known to occur if, e.g., $\alpha > \sqrt{d}\nu C_R^3$ [8], then there exists an N such that $\|\nabla \cdot u_h^k\| \leq tol$ for $k \geq N$. If tol is sufficiently small, then it is reasonable to assume $u_h = u_h^N$.

After convergence, we recover pressure via $p_h := -\sum_{i=0}^{N}(\alpha(\nabla \cdot u_h^i))$, thus satisfying

$$\nu(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = (f, v_h) \ \forall v_h \in X_h. \tag{3.4}$$

Lemma 3.1. *Suppose α is sufficiently large so that Algorithm 3.2 converges, and so that $\frac{{C_R}^2}{\alpha} < \frac{1}{2}$. Then,*

$$\|p_h\| \leq \nu^{-\frac{1}{2}}\|f\|_{-1}.$$

Proof. The initial step of the iterated penalty method is defined in (3.3a). The system is known to be well-posed (and can be proven using the Lax-Milgram Theorem). Choosing $v = u_h^0$ and using the definition of $\|\cdot\|_{-1}$ and Young's inequality gives

$$\nu\|\nabla u_h^0\|^2 + \alpha\|\nabla \cdot u_h^0\|^2 = (f, u_h^0) \leq \frac{\nu^{-1}}{2}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla u_h^0\|^2.$$

Reducing now implies that

$$\frac{\nu}{2}\|\nabla u_h^0\|^2 + \alpha\|\nabla \cdot u_h^0\|^2 \leq \frac{\nu^{-1}}{2}\|f\|_{-1}^2. \tag{3.5}$$

Now, we consider our earlier definition of $p_h := -\sum_{k=0}^{N}(\alpha(\nabla \cdot u_h^k))$, where N is the number of iterations of the iterated penalty method. From previous work in [8], we know that $\|\nabla \cdot u_h^k\| \leq \frac{C_R{}^2}{\alpha}\|\nabla \cdot u_h^{k-1}\| \; \forall k \in \mathbb{N}$. For ease of notation, let $r := \frac{C_R{}^2}{\alpha}$. Then $0 < r < \frac{1}{2} < 1$ (by the assumption that $\frac{C_R^2}{\alpha} < \frac{1}{2}$), and so

$$
\begin{aligned}
\|p_h\| &\leq \alpha \sum_{k=0}^{N} \|\nabla \cdot u_h^k\| \\
&= \alpha(\; \|\nabla \cdot u_h^0\| + r\|\nabla \cdot u_h^0\| + \cdots + r^k\|\nabla \cdot u_h^0\|) \\
&= \alpha\|\nabla \cdot u_h^0\|(1 + r + \cdots + r^N) \\
&= \alpha\|\nabla \cdot u_h^0\| \left(\frac{1 - r^{N+1}}{1 - r}\right) \\
&\leq \alpha\|\nabla \cdot u_h^0\| \left(\frac{1}{1 - \frac{1}{2}}\right) \\
&= 2\alpha\|\nabla \cdot u_h^0\|.
\end{aligned}
$$

Using (3.5), we obtain the desired result,

$$\|p_h\| \leq \nu^{-\frac{1}{2}}\|f\|_{-1}.$$

□

Theorem 3.3. *Let (w_h^γ, q^γ) be the solution to Algorithm 3.1 for a fixed $\gamma \geq 0$ and let u_h be the solution to Algorithm 3.2. Then,*

$$\|\nabla(w_h^\gamma - u_h)\| \leq \gamma^{-1}\nu^{-\frac{1}{2}}C_R^{-1}\|f\|_{-1}.$$

Thus, as $\gamma \to \infty$, the sequence of grad-div stabilized Taylor-Hood velocity solutions, $\{w_h^\gamma\}$, converges to u_h.

Proof. Denote $e := w_h^\gamma - u_h$ and let p_h be the recovered pressure from Algorithm 3.2. We subtract (3.4) from (3.1a) to obtain

$$\nu(\nabla e, \nabla v_h) - (q_h^\gamma - p_h, \nabla \cdot v_h) + \gamma(\nabla \cdot e, \nabla \cdot v_h) = 0 \; \forall v_h \in X_h. \tag{3.6}$$

Orthogonally decompose $e \in V_h$ as $e = e_0 + e_r$ where $e_0 \in V_h^0, e_r \in R_h$. Choosing $v_h = e_0$ in (3.6). This yields $\nu\|\nabla e_0\|^2 = 0$ which implies $e_0 = 0$. Next choose $v_h = e_r$ in (3.6). This gives

$$\nu\|\nabla e_r\|^2 + (p_h, \nabla \cdot e_r) + \gamma\|\nabla \cdot e_r\|^2 = 0,$$

and after applying Cauchy-Schwarz and Young inequalities to the pressure term, we obtain

$$\nu\|\nabla e_r\|^2 + \gamma\|\nabla \cdot e_r\|^2 = -(p_h, \nabla \cdot e_r) \leq \frac{\gamma}{2}\|\nabla \cdot e_r\|^2 + \frac{\gamma^{-1}}{2}\|p_h\|^2.$$

Further reduction yields

$$\|\nabla \cdot e_r\|^2 \leq 2\gamma^{-1}\nu\|\nabla e_r\|^2 + \|\nabla \cdot e_r\|^2 \leq \gamma^{-2}\|p_h\|^2,$$

and after taking the square root of both sides, we obtain

$$\|\nabla \cdot e_r\| \leq \gamma^{-1}\|p_h\|.$$

Using (2.3), we have that

$$\|\nabla e_r\| \leq C_R^{-1}\|\nabla \cdot e_r\| \leq \gamma^{-1}C_R^{-1}\|p_h\|.$$

and after applying Lemma 3.1, we obtain

$$\|\nabla e_r\| \leq \gamma^{-1}\nu^{-\frac{1}{2}}C_R^{-1}\|f\|_{-1}. \tag{3.7}$$

Thus, using $e_0 = 0$ and (3.7), we conclude

$$\|\nabla e\| = \|\nabla e_r\| \le \gamma^{-1}\nu^{-\frac{1}{2}} C_R^{-1} \|f\|_{-1}. \tag{3.8}$$

$\square$

Corollary 3.1. *Under the same assumptions as Theorem 3.3, if we further assume $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition (with parameter β_0), then we obtain convergence of a modified pressure:*

$$\|(q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h\| \le \gamma^{-1}\nu^{\frac{1}{2}}\beta_0^{-1} C_R^{-1}\|f\|_{-1}.$$

Proof. We can rewrite (3.6) as

$$\nu(\nabla e, \nabla v_h) = ((q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h) \ \forall v_h \in X_h.$$

Dividing both sides by $\|\nabla v_h\|$, with $\|\nabla v_h\| \ne 0$, we have $\forall v_h \in X_h$

$$\frac{((q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} = \frac{\nu(\nabla e, \nabla v_h)}{\|\nabla v_h\|}.$$

Taking the infimum over $v_h \in X_h$ and applying the assumed inf-sup condition, we obtain

$$\beta_0\|(q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h\| \le \nu\|\nabla e\|.$$

Finally, applying Theorem 3.3 yields

$$\|(q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h\| \le \gamma^{-1}\nu^{\frac{1}{2}}\beta_0^{-1} C_R^{-1}\|f\|_{-1}.$$

$\square$

3.1 Numerical Results

We now test the theory above on both barycenter and uniform meshes on $\Omega = (0,1)^2$ with $\frac{1}{h} = 16$ using (P_2, P_1) elements. On uniform meshes, LBB on the spaces is not known to hold. In this case, we find convergence of velocity, but no pressure convergence. On barycenter meshes with P_2 elements, however, it is known that $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition, so in this case we expect convergence of both velocity and pressure.

We chose the true solution to be

$$u_{true} = \begin{pmatrix} \cos(y) \\ \sin(x) \end{pmatrix},$$
$$p_{true} = \sin(x + y).$$

We enforce Dirichlet velocity boundary conditions to be the true solution at the boundary, set $\nu = 0.01$, and calculate the forcing using

$$f = -\nu\triangle u_{true} + \nabla p_{true}.$$

Solutions were computed using Algorithm 3.1 with varying γ and Algorithm 3.2 (using $\alpha = 1000$).

Table 1 shows results using a barycenter mesh, and here we observe convergence of both velocity and the modified pressure. Results using a uniform mesh are shown in Table 2. Here we observe $O(\gamma^{-1})$ convergence of the velocity but no convergence of the modified pressure.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate
0	2.354E-02		2.676E-04	
10^{-1}	2.844E-03		4.803E-05	
10^0	3.558E-04	0.90	6.877E-06	0.84
10^1	3.671E-05	0.99	7.215E-07	0.98
10^2	3.684E-06	1.00	7.251E-08	1.00
10^3	3.686E-07	1.00	7.266E-09	1.00
10^4	4.864E-08	0.88	1.825E-09	0.60

Table 1: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a barycenter mesh.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate
0	1.290E-03		1.458E-03	
10^{-1}	2.529E-04		1.457E-03	
10^0	1.845E-04	0.14	1.457E-03	0.00
10^1	8.740E-05	0.32	1.457E-03	0.00
10^2	1.885E-05	0.67	1.457E-03	0.00
10^3	2.212E-06	0.93	1.457E-03	0.00
10^4	2.183E-07	1.01	1.457E-03	0.00

Table 2: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a uniform mesh.

4 A connection for the Oseen equations

We next consider the Oseen equations, which are given by:

$$\begin{aligned} \sigma u + U\cdot\nabla u - \nu\triangle u + \nabla p &= f, \\ \nabla\cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

where u represents velocity, p represents pressure, f is an external forcing, σ can be considered as either a friction coefficient or inversely proportional to a time step if f is appropriately modified (e.g. for backward Euler $\sigma = \frac{1}{\triangle t}$ and $f = f + \frac{1}{\triangle t}u^n$, with u^n being the solution at the previous time step), ν is the kinematic viscosity, and $U \in H'(\Omega)$ is given. We will compare the grad-div stabilized Taylor-Hood scheme and the pointwise divergence-free scheme to determine a connection between the two methods. These two methods are given below:

Algorithm 4.1. *Given $\gamma \geq 0$, find $(w_h^\gamma, q_h^\gamma) \in (X_h, Q_h)$ satisfying*

$$\sigma(w_h^\gamma, v_h) + \nu(\nabla w_h^\gamma, \nabla v_h) + b^*(U, w_h^\gamma, v_h) - (q_h^\gamma, \nabla\cdot v_h) + \gamma(\nabla\cdot w_h^\gamma, \nabla\cdot v_h) = (f, v_h)\ \forall v_h \in X_h \tag{4.1a}$$

$$(\nabla\cdot w_h^\gamma, r_h) = 0 \qquad \forall r_h \in Q_h \tag{4.1b}$$

Algorithm 4.2. *Find $u_h \in V_h^0$ satisfying*

$$\sigma(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + b^*(U, u_h, v_h) = (f, v_h)\ \forall v_h \in V_h^0. \tag{4.2}$$

We use the iterated penalty method to solve (4.2)*:*
Step 0: Find $u_h^0 \in X_h$ satisfying

$$\sigma(u_h^0, v_h) + \nu(\nabla u_h^0, \nabla v_h) + b^*(U, u_h^0, v_h) + (\alpha(\nabla\cdot u_h^0), \nabla\cdot v_h) = (f, v_h)\ \forall v_h \in X_h \tag{4.3}$$

Step k: Find $u_h^k \in X_h$ satisfying

$$\sigma(u_h^k, v_h) + \nu(\nabla u_h^k, \nabla v_h) + b^*(U, u_h^k, v_h) + (\alpha(\nabla \cdot u_h^k), \nabla \cdot v_h) = \sigma(u_h^{k-1}, v_h) + \nu(\nabla u_h^{k-1}, \nabla v_h) + b^*(U, u_h^{k-1}, v_h) \ \forall v_h \in X_h \tag{4.4}$$

If the iterated penalty method converges, which is known to occur when $\alpha > \sqrt{d}\nu C_R^3$ [8], then there exists an N such that $\|\nabla \cdot u_h^k\| \leq tol$ for $k \geq N$. If tol is sufficiently small, then it is reasonable to assume $u_h = u_h^N$.

After convergence, we recover pressure via $p_h := -\sum_{i=0}^{N}(\alpha(\nabla \cdot u_h^i))$ to satisfy

$$\sigma(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + b^*(U, u_h, v_h) - (p_h, \nabla \cdot v_h) = (f, v_h) \ \forall v_h \in X_h. \tag{4.5}$$

Lemma 4.1. *Suppose α is sufficiently large so that Algorithm 4.2 converges (e.g. if $\alpha > \sqrt{d}\nu C_R^3$), and so that $\frac{C}{\alpha} < \frac{1}{2}$, where C is a data-dependent constant which satisfies $\|\nabla \cdot u_h^k\| \leq \frac{C}{\alpha}\|\nabla \cdot u_h^{k-1}\|$ (existence of such a constant is shown in [8]). Then,*

$$\|p_h\| \leq \nu^{-\frac{1}{2}}\|f\|_{-1}.$$

Proof. The system is known to be well-posed and can be proven (by the Lax-Milgram Theorem). Choosing $v = u_h^0$ in (4.3) and using the definition of $\|\cdot\|_{-1}$ and the Cauchy-Schwarz and Young's inequalities gives

$$\sigma\|u_h^0\|^2 + \nu\|\nabla u_h^0\|^2 + \alpha\|\nabla \cdot u_h^0\|^2 = (f, u_h^0) \leq \frac{\nu^{-\frac{1}{2}}}{2}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla u_h^0\|^2.$$

Reducing implies that

$$\sigma\|u_h^0\|^2 + \frac{\nu}{2}\|\nabla u_h^0\|^2 + \alpha\|\nabla \cdot u_h^0\|^2 \leq \frac{\nu^{-\frac{1}{2}}}{2}\|f\|_{-1}^2. \tag{4.6}$$

Now, we consider our earlier definition of $p_h := -\sum_{k=0}^{N}(\alpha(\nabla \cdot u_h^k))$, where N is the number of steps until convergence (N is guaranteed to be finite since α is chosen to be sufficiently large). From [8], we know that $\exists C$ such that

$$\|\nabla \cdot u_h^k\| \leq \frac{C}{\alpha}\|\nabla \cdot u_h^{k-1}\|.$$

For ease of notation, let $r := \frac{C}{\alpha}$. This yields

$$\|p_h\| \leq \alpha\sum_{k=0}^{N}\|\nabla \cdot u_h^k\| = \alpha\|\nabla \cdot u_h^0\|\left(\frac{1 - r^{N+1}}{1 - r}\right).$$

Appyling (4.6) gives

$$\|p_h\| \leq \frac{\nu^{-\frac{1}{2}}}{2}\|f\|_{-1}\left(\frac{\alpha}{\alpha - C}\right).$$

From the assumptions on C and α, we can further reduce to obtain

$$\|p_h\| \leq \nu^{-\frac{1}{2}}\|f\|_{-1}.$$

□

Theorem 4.3. *Let (w_h^γ, q^γ) be the solution to Algorithm 4.1 for a fixed $\gamma \geq 0$, let u_h be the solution to Algorithm 4.2. Then,*

$$\|\nabla(w_h^\gamma - u_h)\| \leq \gamma^{-1}\frac{\nu^{-1/2}}{2}(1 + MC_U)C_R^{-1}\|f\|_{-1}.$$

Thus, as $\gamma \to \infty$, the sequence of grad-div stabilized Taylor-Hood velocity solutions, $\{w_h^\gamma\}$, converges to u_h.

Proof. Denote $e := w_h^\gamma - u_h$, and let p_h be the recovered pressure from Algorithm 4.2. Subtracting (4.5) from (4.1a), we have

$$\sigma(e, v_h) + \nu(\nabla e, \nabla v_h) + b^*(U, e, v_h) + \gamma(\nabla \cdot e, \nabla \cdot v_h) - (q_h^\gamma - p_h, \nabla \cdot v_h) = 0 \ \forall v_h \in X_h. \tag{4.7}$$

Choose $v_h = e$ in (4.7) to obtain

$$\sigma\|e\|^2 + \nu\|\nabla e\|^2 + \gamma\|\nabla \cdot e\|^2 = -(q_h^\gamma - p_h, \nabla \cdot e).$$

Now, orthogonally decompose $e \in V_h$ as $e = e_0 + e_r$ where $e_0 \in V_h^0, e_r \in R_h$ and apply Cauchy-Schwarz and Young inequalities to the pressure term (which reduces because $e \in V_h$). This yields

$$\sigma\|e_r\|^2 + \nu\|\nabla e_r\|^2 + \gamma\|\nabla \cdot e_r\|^2 \leq \frac{\gamma^{-1}}{2}\|p_h\|^2 + \frac{\gamma}{2}\|\nabla \cdot e_r\|^2,$$

which implies

$$\gamma\|\nabla \cdot e_r\|^2 \leq \gamma^{-1}\|p_h\|^2.$$

Applying Lemma 4.1 and taking γ to the other side of the equation, we obtain

$$\|\nabla \cdot e_r\|^2 \leq \gamma^{-2}\frac{\nu^{-1}}{2}\|f\|_{-1}^2.$$

Taking square roots and using (2.3), we have

$$\|\nabla e_r\| \leq \gamma^{-1}\frac{\nu^{-\frac{1}{2}}}{2}\|f\|_{-1}C_R^{-1}. \tag{4.8}$$

The next step is to bound $\|\nabla e_0\|$. Choose $v_h = e_0$ in (4.7). As, $b^*(U, e, e_0) = b^*(U, e_r, e_0)$, this gives

$$\sigma\|e_0\|^2 + \nu\|\nabla e_0\|^2 + b^*(U, e_r, e_0) = 0,$$

and after taking the nonlinear term to the other side, we obtain

$$\sigma\|e_0\|^2 + \nu\|\nabla e_0\|^2 \leq \|b^*(U, e_r, e_0)\|.$$

Using (2.2) and the assumptions on U, we have

$$\sigma\|e_0\|^2 + \nu\|\nabla e_0\|^2 \leq MC_U\|\nabla e_r\|\|\nabla e_0\|,$$

which further implies

$$\nu\|\nabla e_0\| \leq MC_U\|\nabla e_r\|. \tag{4.9}$$

Thus, using (4.8) and (4.9), we can conclude that

$$\begin{aligned}\|\nabla e\| = \|\nabla e_0\| + \|\nabla e_r\| &\leq (1 + MC_U)\|\nabla e_r\| \\ &\leq \gamma^{-1}\frac{\nu^{-1/2}}{2}(1 + MC_U)C_R^{-1}\|f\|_{-1}.\end{aligned}$$

□

Corollary 4.1. *Under the same assumptions as Theorem 4.3, if we further assume $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition (with parameter β_0), then we obtain convergence of a modified pressure:*

$$\|(q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h\| \leq \gamma^{-1}\frac{\nu^{-\frac{1}{2}}}{2}\beta_0^{-1}C_R^{-1}\|f\|_{-1}(1 + MC_U)(\sigma C_{PF} + \nu + MC_U).$$

Proof. Rewriting (4.7) yields

$$((q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h) = \sigma(e, v_h) + \nu(\nabla e, \nabla v_h) + b^*(U, e, v_h)|\|\nabla v_h\|$$

Taking the pressure term to the other side and dividing both sides by $\|\nabla v_h\|$, with $\|\nabla v_h\| \neq 0$, we have $\forall v_h \in X_h$,

$$\frac{((q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} = \frac{|\sigma(e, v_h) + \nu(\nabla e, \nabla v_h) + b^*(U, e, v_h)|}{\|\nabla v_h\|}.$$

Applying (**??**), we have

$$\frac{((q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq \frac{\sigma(e, v_h) + \nu(\nabla e, \nabla v_h)}{\|\nabla v_h\|} + \frac{MC_U \|\nabla e\| \|\nabla v_h\|}{\|\nabla v_h\|}.$$

which simplifies to

$$\frac{((q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq \frac{\sigma(e, v_h) + \nu(\nabla e, \nabla v_h)}{\|\nabla v_h\|} + MC_U \|\nabla e\|.$$

Taking the infimum over $v_h \in X_h$ and applying the assumed inf-sup condition, we obtain

$$\beta_0 \|(q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h\| \leq (\sigma C_{PF} \|e\| + \nu \|\nabla e\| + MC_U \|\nabla e\|).$$

Using (2.1), we have

$$\|(q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h\| \leq \beta_0^{-1} (\sigma C_{PF} + \nu + MC_U) \|\nabla e\|.$$

After applying Theorem 4.3 , we have

$$\|(q_h^\gamma - \gamma \nabla \cdot w_h^\gamma) - p_h\| \leq \gamma^{-1} \frac{\nu^{-\frac{1}{2}}}{2} \beta_0^{-1} C_R^{-1} \|f\|_{-1} (1 + MC_U)(\sigma C_{PF} + \nu + MC_U).$$

□

4.1 Numerical Results

We now test the theory above on both uniform and barycenter meshes of $\Omega = (0,1)^2$ with $\frac{1}{h} = 16$ using (P_2, P_1) elements. On barycenter meshes with P_2 elements, it is known that $(X_h, \nabla \cdot X_h)$ is LBB stable, so in this case we expect convergence of both velocity and pressure. On uniform meshes, however, LBB on the spaces is not known to hold. In this case, we find convergence of velocity, but no pressure convergence.

We again chose the true solution to be

$$u_{true} = \begin{pmatrix} \cos(y) \\ \sin(x) \end{pmatrix},$$
$$p_{true} = \sin(x + y).$$

We enforce Dirichlet velocity boundary conditions to be the true solution at the boundary, $\sigma = 0.1$, $\nu = 0.01$, $U = u_{true}$, and calculated the forcing using the true solution and

$$f = \sigma u_{true} + U \cdot \nabla u_{true} - \nu \triangle u_{true} + \nabla p_{true}.$$

Solutions were computed using Algorithm 4.1 with varying γ and Algorithm 4.2 (using $\alpha = 1000$).

Results using a barycenter mesh are shown in Table 3. Here we observe $O(\gamma^{-1})$ convergence of both velocity and the modified pressure. Table 4 shows results using a uniform mesh, and we observe convergence of velocity but no convergence of the modified pressure.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate
0	2.236E+00		2.723E-02	
10^{-1}	2.881E-01		5.271E-03	
10^0	3.698E-02	0.89	7.813E-04	0.83
10^1	3.834E-03	0.98	8.246E-05	0.98
10^2	3.849E-04	1.00	8.292E-06	1.00
10^3	3.850E-05	1.00	8.297E-07	1.00
10^4	3.850E-06	1.00	8.308E-08	1.00

Table 3: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a barycenter mesh.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate
0	1.282E-01		1.458E-01	
10^{-1}	2.522E-02		1.458E-01	
10^0	1.840E-02	0.14	1.457E-01	0.00
10^1	8.735E-03	0.32	1.457E-01	0.00
10^2	1.882E-03	0.67	1.457E-01	0.00
10^3	2.214E-04	0.93	1.457E-01	0.00
10^4	2.254E-05	0.99	1.457E-01	0.00

Table 4: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a uniform mesh.

5 A connection for the Boussinesq equations

We now consider the Boussinesq equations for the flow of heated silicon oil, which are given by:

$$\begin{aligned}
-\triangle u + \nabla p &= R_a \begin{pmatrix} 0 \\ T \end{pmatrix}, \\
\nabla \cdot u &= 0, \\
-\triangle T + u \cdot \nabla T &= g, \\
u|_{\partial\Omega} &= 0,
\end{aligned}$$

where u represents velocity, p represents pressure, T represents temperature, f is an external forcing, g encompasses the Dirichlet boundary conditions on the temperature, and R_a is the Rayleigh number.

We will compare grad-div stabilized Taylor-Hood and the pointwise divergence-free solution to determine a connection between the two methods. The two methods are given below:

Algorithm 5.1. *Given $\gamma \geq 0$, find $(w_h^\gamma, q_h^\gamma, \Lambda_h^\gamma) \in (X_h, Q_h, S_h)$ satisfying*

$$(\nabla w_h, \nabla v_h) - (q_h, \nabla \cdot v_h) + \gamma(\nabla \cdot w_h, \nabla \cdot v_h) = R_a \left(\begin{pmatrix} 0 \\ \Lambda_h^\gamma \end{pmatrix}, v_h \right) \; \forall v_h \in X_h, \tag{5.1a}$$

$$(\nabla \cdot w_h, r_h) = 0 \qquad \forall r_h \in Q_h, \tag{5.1b}$$

$$(\nabla \Lambda_h^\gamma, \nabla s_h) + b^*(w_h, \Lambda_h^\gamma, s_h) = (g, s_h) \qquad \forall s_h \in S_h. \tag{5.1c}$$

Algorithm 5.2. *Find $(u_h, T_h) \in (V_h^0, S_h)$ satisfying*

$$(\nabla u_h, \nabla v_h) = R_a \left(\begin{pmatrix} 0 \\ T_h \end{pmatrix}, v_h \right) \; \forall v_h \in V_h^0, \tag{5.2a}$$

$$(\nabla T_h, \nabla s_h) + b^*(u_h, T_h, s_h) = (g, s_h) \qquad \forall s_h \in S_h. \tag{5.2b}$$

We use an iterated-penalty-quasi-Newton method to solve (5.2a) - (5.2b)*:*

Step 0: Find (u_k^0, T_k^0) *where*

$$(\nabla u_k^0, \nabla v_h) + \alpha(\nabla \cdot u_k^0, \nabla \cdot v_h) - R_a\left(\begin{pmatrix} 0 \\ T_k \end{pmatrix}, v_h\right) = 0 \ \forall v_h \in X_h, \tag{5.3a}$$

$$(\nabla T_k^0, \nabla s_h) + b^*(u_k^0, T_{k-1}, s_h) + b^*(u_{k-1}, T_k^0, s_h) - b^*(u_{k-1}, T_{k-1}, s_h) = 0 \ \forall s_h \in S_h, \tag{5.3b}$$

where u_{k-1} *and* T_{k-1} *are the initial guesses (typically 0).*

Step n: Find (u_k^n, T_k^n) *where*

$$(\nabla u_k^n, \nabla v_h) + \alpha(\nabla \cdot u_k^n, \nabla \cdot v_h) - R_a\left(\begin{pmatrix} 0 \\ T_k^n \end{pmatrix}, v_h\right) = (\nabla u_k^{n-1}, \nabla v_h) + \alpha(\nabla \cdot u_k^{n-1}, \nabla \cdot v_h) \ \forall v_h \in X_h, \tag{5.4a}$$

$$(\nabla T_k^n, \nabla s_h) + b^*(u_k^n, T_{k-1}, s_h) + b^*(u_{k-1}, T_k^n, s_h) - b^*(u_{k-1}, T_{k-1}, s_h) = 0 \ \forall s_h \in S_h, \tag{5.4b}$$

where u_{k-1} *and* T_{k-1} *are the solutions to the previous step.*

Assuming that the iterated penalty method converges, then there exists an N *such that* $\|\nabla \cdot u_h^k\| \leq tol$ *for* $k \geq N$*. If tol is sufficiently small, it is reasonable to assume* $u_h = u_h^N$*. In our computation, we choose* $tol = 10^{-9}$*. We end the outer iteration when* $\|u_k - u_{k-1}\| < tol$*.*

After convergence, we recover pressure via $p_h := -\sum_{i=0}^{k}(\alpha(\nabla \cdot u_h^i))$ *to satisfy*

$$(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = R_a\left(\begin{pmatrix} 0 \\ T_h \end{pmatrix}, v_h\right) \ \forall v_h \in X_h, \tag{5.5a}$$

$$(\nabla T_h, \nabla s_h) + b^*(u_h, T_h, s_h) = (g, s_h) \qquad \forall s_h \in S_h. \tag{5.5b}$$

Remark 5.3. *Throughout this section, we use the small data condition:*

$$R_a C_{PF}^2 C_g M < 1.$$

A small data condition is also used in [2].

Lemma 5.1. *If the data satisfies the small data condition* $R_a C_{PF}^2 C_g M < 1$*, then solutions to Algorithm 5.1 exist uniquely.*

Proof. Existence of solutions can be proven using the exact same techniques as in [2]. For uniqueness, assume two solutions, say (w_1, q_1, Λ_1) and (w_2, q_2, Λ_2). Note

$$\|\nabla \Lambda_2\|^2 = (g, \Lambda_2).$$

Denote $e_w := w_1 - w_2$, $e_q := q_1 - q_2$, and $e_\Lambda := \Lambda_1 - \Lambda_2$. We subtract to obtain

$$\begin{aligned}
(\nabla e_w, \nabla v) - (e_q, \nabla \cdot v) + \gamma(\nabla \cdot e_w, \nabla \cdot v) &= R_a\left(\begin{pmatrix} 0 \\ e_\Lambda \end{pmatrix}, v\right) && \forall v \in X, \\
(\nabla \cdot e_w, r) &= 0 && \forall r \in Q, \\
(\nabla e_\Lambda, \nabla s) + b^*(w_1, e_\Lambda, s) + b^*(e_w, \Lambda_2, s) &= (g, s) && \forall s \in S.
\end{aligned}$$

Choose $v = e_w$, $r = e_q$, and $s = e_\Lambda$. Then

$$\|\nabla e_w\|^2 + \gamma\|\nabla e_w\| = R_a\left(\begin{pmatrix} 0 \\ e_\Lambda \end{pmatrix}, v\right) \tag{5.6a}$$

$$\leq R_a\|\nabla e_\Lambda\|\|e_w\| \tag{5.6b}$$

$$\leq \frac{1}{2}\|\nabla e_w\|^2 + \frac{C_{PF}^4 R_a^2}{2}\|\nabla e_\Lambda\|^2 \tag{5.6c}$$

and

$$\|\nabla e_\Lambda\|^2 + b^*(e_w, \Lambda_2, e_\Lambda) = 0. \tag{5.7}$$

Further reducing (5.7) yields

$$\|\nabla e_\Lambda\| \le M\|\nabla e_w\|\|\nabla \Lambda_2\| \tag{5.8a}$$

$$\le MC_g\|\nabla e_w\| \tag{5.8b}$$

Combining (5.6c) and (5.8b), we obtain

$$\|\nabla e_w\|^2 \le C_{PF}^4 R_a^2 M^2 C_g^2 \|\nabla e_w\|^2.$$

Thus, uniqueness holds if $C_{PF}^4 R_a^2 M^2 C_g^2 < 1$, i.e. if the small data condition above holds.

□

We will continue with an a priori bound on Λ_h^γ.

Lemma 5.2. *Suppose the data satisfies the small data condition $R_a C_{PF}^2 C_g M < 1$ so that Algorithm 5.1 has a unique solution $(w_h^\gamma, q_h^\gamma, \Lambda_h^\gamma)$. Then*

$$\|\nabla \Lambda_h^\gamma\| \le C_g,$$

where C_g depends only on g.

Proof. Let $s_h = \Lambda_h^\gamma$ in (5.1c). Then we have

$$\|\nabla \Lambda_h^\gamma\|^2 = (g, \Lambda_h^\gamma).$$

Applying (2.1) and the Cauchy-Schwarz inequality, we obtain

$$\|\nabla \Lambda_h^\gamma\|^2 \le C_{PF}\|g\|_{-1}\|\nabla \Lambda_h^\gamma\|.$$

Simplifying, we are left with

$$\|\nabla \Lambda_h^\gamma\| \le C_{PF}\|g\|_{-1} = C_g.$$

□

Theorem 5.4. *Suppose the data is sufficient to allow for Algorithm 5.1 and Algorithm 5.2 to have unique solutions, $(w_h^\gamma, q_h^\gamma, \Lambda_h^\gamma)$ and (u_h, T_h) respectively. Let p_h be the pressure recovered from Algorithm 5.2. Further, suppose $\|p_h\| \le C_p < \infty$ and the data satisfies $R_a C_{PF}^2 C_g M < 1$. Then,*

$$\|\nabla(w_h^\gamma - u_h)\| \le \gamma^{-1} C_R^{-1} C_p,$$

and

$$\|\nabla(\Lambda_h^\gamma - T_h)\| \le \gamma^{-1} C_R^{-1} C_p M C_g.$$

Thus, as $\gamma \to \infty$, the sequence of grad-div stabilized Taylor-Hood velocity solutions $\{w_h^\gamma\}$ converges to u_h and the sequence of temperature solutions $\{\Lambda_h^\gamma\}$ converges to T_h.

Proof. We subtract (5.5a) from (5.1a) to obtain (5.9a) and (5.5b) from (5.1c) to obtain (5.9b). Denoting $e^u := w_h - u_h$ and $e^T := \Lambda_h^\gamma - T_h$, we find

$$(\nabla e^u, \nabla v_h) - (q_h^\gamma - p_h, \nabla \cdot v_h) + \gamma(\nabla \cdot e^u, \nabla \cdot v_h) = R_a \left(\begin{pmatrix} 0 \\ e^T \end{pmatrix}, v_h \right) \quad \forall v_h \in X_h, \tag{5.9a}$$

$$(\nabla e^T, \nabla s_h) + b^*(u_h, e^T, s) + b^*(e^u, \Lambda_h^\gamma, s) = 0 \quad \forall s_h \in S_h. \tag{5.9b}$$

Choose $s_h = e^T$ in (5.9b). Then

$$\|\nabla e^T\|^2 + b^*(e^u, \Lambda_h^\gamma, e^T) = 0.$$

Taking the b^* term to the other side and applying (2.2), we obtain

$$\|\nabla e^T\|^2 \leq M\|\nabla e^u\|\|\nabla \Lambda_h^\gamma\|\|\nabla e^T\|.$$

After simplifying and using Lemma 5.2, we are left with

$$\|\nabla e^T\| \leq MC_g\|\nabla e^u\|. \tag{5.10}$$

Next, we choose $v_h = e^u$ in (5.9a), which yields

$$\|\nabla e^u\|^2 + \gamma\|\nabla \cdot e^u\|^2 = (q_h - p_h, \nabla \cdot e^u) + R_a\left(\begin{pmatrix} 0 \\ e^T \end{pmatrix}, e^u\right).$$

Using Cauchy-Schwarz and Young's inequalities, (2.1), (5.1b), and Lemma 5.2, we obtain

$$\begin{aligned}
\|\nabla e^u\|^2 + \gamma\|\nabla \cdot e^u\|^2 &\leq \|p_h\|\|\nabla \cdot e^u\| + R_a\|e^T\|\|e^u\| \\
&\leq \frac{\gamma^{-1}}{2}\|p_h\|^2 + \frac{\gamma}{2}\|\nabla \cdot e^u\|^2 + R_aC_{PF}^2\|\nabla e^T\|\|\nabla e^u\| \\
&= \frac{\gamma^{-1}}{2}\|p_h\|^2 + \frac{\gamma}{2}\|\nabla \cdot e^u\|^2 + R_aC_{PF}^2C_gM\|\nabla e^u\|,
\end{aligned}$$

which further yields

$$\|\nabla e^u\|^2(1 - R_aC_{PF}^2C_gM) + \frac{\gamma}{2}\|\nabla \cdot e^u\|^2 \leq \frac{\gamma^{-1}}{2}\|p_h\|^2.$$

Thus, we obtain

$$\|\nabla \cdot e^u\| \leq \gamma^{-1}\|p_h\|,$$

and using (2.3) and the assumption on p_h,

$$\|\nabla e^u\| \leq \gamma^{-1}C_R^{-1}C_p.$$

Furthermore, from (5.10),

$$\|\nabla e^T\| \leq \gamma^{-1}C_R^{-1}C_pMC_g.$$

□

Corollary 5.1. *Under the same assumptions as Theorem 5.4, if we further assume $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition (with parameter β_0), then we obtain convergence of a modified pressure:*

$$\|(q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h\| \leq \gamma^{-1}C_R^{-1}C_p(1 + R_aC_{PF}^2C_gM)\beta_0^{-1}.$$

Proof. Equation (5.9a) can be rewritten as

$$(\nabla e^u, \nabla v_h) - (q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h) = R_a\left(\begin{pmatrix} 0 \\ e^T \end{pmatrix}, v_h\right) \;\; \forall v_h \in X_h,$$

and after rearranging and dividing both sides by $\|\nabla v_h\|$, with $\|\nabla v_h\| \neq 0$, we have $\forall v_h \in X_h$,

$$\begin{aligned}
\frac{((q_h^\gamma - \gamma\nabla \cdot w_h^\gamma) - p_h, \nabla \cdot v_h)}{\|\nabla v_h\|} &= \frac{\nu(\nabla e, \nabla v_h) - R_a((\begin{smallmatrix} 0 \\ e^T \end{smallmatrix}), v_h)}{\|\nabla v_h\|} \\
&\leq \frac{\|\nabla e^u\|\|\nabla v_h\| + R_a\|e^T\|\|v_h\|}{\|\nabla v_h\|} \\
&\leq \|\nabla e^u\| + R_aC_{PF}^2\|\nabla e^T\|.
\end{aligned}$$

Using (5.10) and then Theorem 5.4, we obtain

$$\frac{((q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h, \nabla\cdot v_h)}{\|\nabla v_h\|} \le \|\nabla e^u\|(1 + R_a C_{PF}^2 C_g M)$$
$$\le \gamma^{-1} C_R^{-1} C_p (1 + R_a C_{PF}^2 C_g M).$$

Taking the infimum over $v_h \in X_h$ and applying the assumed inf-sup condition, we obtain

$$\beta_0 \|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\| \le \gamma^{-1} C_R^{-1} C_p (1 + R_a C_{PF}^2 C_g M).$$

This finishes the proof. □

5.1 Numerical Results

The test problem we consider models a heated cavity of silicon oil using $\Omega = (0,1)^2$ and $R_a = 10^5$. We enforce $u|_{\partial\Omega} = 0$, and for temperature we set $T = 1$ on the right side of the box, $T = 0$ on the left side of the box, and weakly enforce the top and bottom be insulated via $\nabla T \cdot n = 0$. Since, we explicitly enforce $T = 1$ on the left side, take $g = 0$. We test the theory on both barycenter and uniform meshes with $\frac{1}{h} = 16$, using (P_2, P_1, P_2) elements. On barycenter meshes with P_2 elements, it is known that $(X_h, \nabla\cdot X_h)$ satisfies the LBB condition, so in this case we expect convergence of velocity, temperature, and modified pressure. On uniform meshes, however, LBB on these spaces is not known to hold. In this case, we find convergence of velocity and temperature, but no modified pressure convergence. Solutions were computed using Algorithm 5.1 with varying γ and Algorithm 5.2 (using $\alpha = 1000$).

Results using a barycenter mesh are shown in Table 5. Here we observe $O(\gamma^{-1})$ convergence of the velocity, pressure, and temperature. In Figure 1, we see the sequence of grad-div stabilized Taylor-Hood solutions converging to the pointwise divergence-free solution. Table 6 shows results using a uniform mesh, and here we observe convergence of velocity and temperature, but no pressure convergence. In Figure 2, we see the sequence of grad-div stabilized Taylor-Hood velocity and temperature solutions converging to the pointwise divergence-free solution, but the pressure solution does not converge.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate	$\|\nabla(\Lambda_h^\gamma - T_h)\|$	Rate
0	4.766E+00		1.539E+01		1.442E-03	
10^{-1}	4.680E+00		1.524E+01		1.424E-03	
10^{0}	4.166E+00	0.05	1.412E+01	0.03	1.283E-03	0.05
10^{1}	2.382E+00	0.24	8.443E+00	0.22	6.957E-04	0.27
10^{2}	4.773E-01	0.70	1.713E+00	0.69	1.311E-04	0.72
10^{3}	5.318E-02	0.95	1.912E-01	0.95	1.443E-05	0.96
10^{4}	5.379E-03	0.99	1.935E-02	0.99	1.457E-06	1.00

Table 5: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a barycenter mesh.

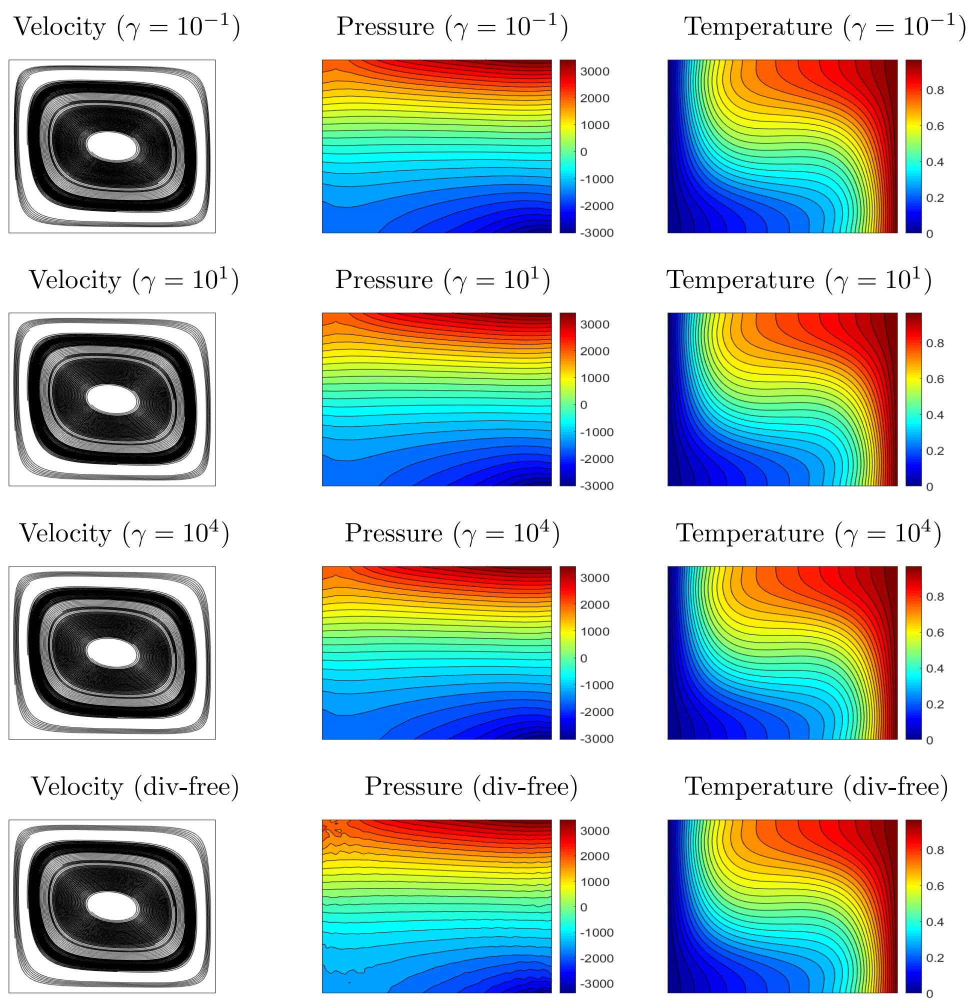

Figure 1: Visuals of three grad-div stabilized Taylor-Hood velocity, pressure, and temperature solutions followed by the pointwise divergence-free solutions, using a barycenter mesh.

γ	$\|\nabla(w_h^\gamma - u_h)\|$	Rate	$\|(q_h^\gamma - \gamma\nabla\cdot w_h^\gamma) - p_h\|$	Rate	$\|\nabla(\Lambda_h^\gamma - T_h)\|$	Rate
0	2.214E+01		5.041E+02		2.426E-02	
10^{-1}	2.213E+01		5.040E+02		2.424E-02	
10^0	2.206E+01	0.00	5.032E+02	0.00	2.413E-02	0.00
10^1	2.146E+01	0.01	4.948E+02	0.01	2.319E-02	0.02
10^2	1.759E+01	0.09	4.296E+02	0.06	1.822E-02	0.10
10^3	7.259E+00	0.38	2.294E+02	0.27	6.952E-03	0.42
10^4	1.302E+00	0.75	9.481E+01	0.38	1.107E-03	0.80
10^5	1.465E-01	0.95	8.642E+01	0.04	1.204E-04	0.96
10^6	1.485E-02	0.99	8.650E+01	0.00	1.215E-05	1.00
10^7	1.487E-03	1.00	8.327E+01	0.02	1.215E-06	1.00

Table 6: Differences between grad-div stabilized Taylor-Hood solutions and pointwise divergence-free solutions on a uniform mesh.

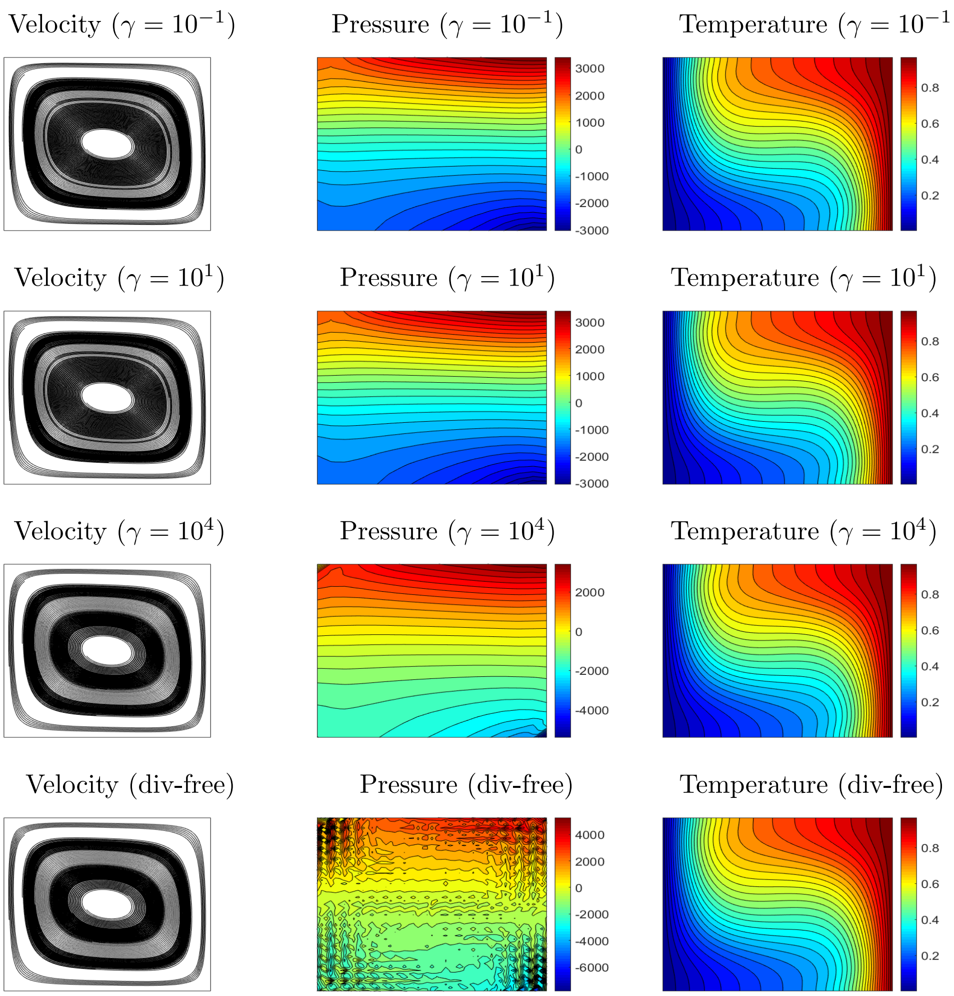

Figure 2: Visuals of three grad-div stabilized Taylor-Hood velocity, pressure, and temperature solutions followed by the pointwise divergence-free solutions, using a uniform mesh.

6 Conclusions

We have proven that for Stokes, Oseen, and Boussinesq systems, grad-div stabilized Taylor-Hood velocity solutions (and temperature solutions for Boussinesq) converge to the pointwise divergence-free solution at a rate of γ^{-1} as $\gamma \to \infty$. Furthermore, if $(X_h, \nabla \cdot X_h)$ satisfies the LBB condition, where X_h is the finite element velocity space, then a modified pressure solution will also converge at the same rate. We verified these results by testing on a barycenter mesh, where this LBB condition is satified, and on a uniform mesh, where it is not known to be satisfied. The numerical results were consistent with the theory in all tests.

Thus, our work generalizes the work of [7, 1], which proved similar results, but only for special meshes having specific macroelement structure.

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