Symmetric Random Walks on Wheel-and-Spokes Graphs

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Abstract

We study symmetric random walks on the vertices of a wheel-and-spokes graph. We consider the following questions. How long does it take for the walk to go from one vertex to another? Starting from one vertex, how long does it take to visit all vertices? Having visited all vertices, how much additional time does it take to return to the starting vertex? The answers to these questions are random variables for which we desire the exact probability distributions, if possible; otherwise, we seek at least their means and standard deviations. We compare our results to those of symmetric random walks on the vertices of polygons.

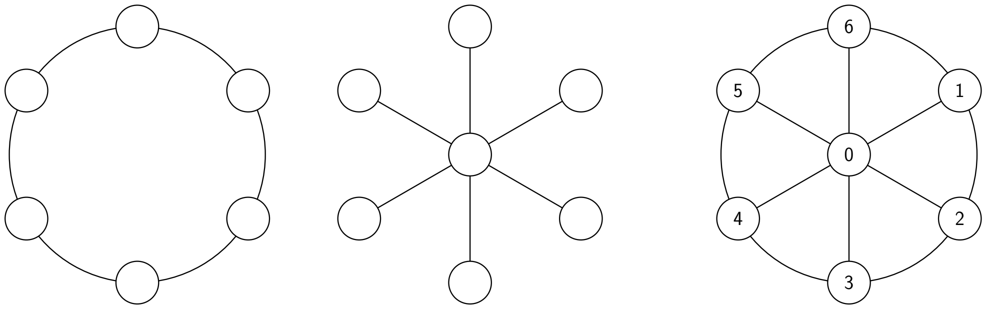
Keywords Stochastic Processes, Absorbing Markov Chain, One-Step Transition Time, Recursive Relations, Cover Time
Subject Classification 05C81; 97K60

1 Introduction

Many physical phenomena that evolve over time can be modeled by random walks on a suitable graph $G = (V, E)$ whose set of vertices V denotes the different states, and set of edges E denotes all possible transitions from one state to another. By a random walk we mean a discrete time stochastic process in which a particle starts from one vertex, called the origin, and at each successive epoch it moves from its current position to an adjacent vertex. Hence, the expressions 'how much time' and 'how many steps' mean the same thing. The walk is called *symmetric* when starting from any vertex, the next transition takes the particle to any one of the adjacent vertices *with equal probability.*

Maiti and Sarkar (2019) studied symmetric random walks (SRW) on the n vertices of a linear path (with absorbing, sticky or reflecting end vertices(s)), and Sarkar (2006) studied SRW on the n vertices of a polygon or on the n nodes on a circle C_n. Here we study a SRW on the vertices of a Wheel-and-Spokes graph, which adds a central vertex to C_n and connects it to each node on the circle. Thus, a Wheel-and-Spokes graph $W_n = (V, E)$ consist of $(n + 1)$ vertices of which n vertices are on a circle and one vertex is at the center. There are $2n$ edges: The center (or hub) is connected to each vertex on the circle (or periphery) resulting in n spokes; each vertex on the circle is also connected with its two neighbors in the clockwise and the counterclockwise directions, thereby constituting a circle graph C_n, which forms the wheel.

Let us label the vertices along the circle (or wheel) as $v_1, v_2, \ldots, v_n$, and the center (or hub) as v_0 (sometimes also labeled as H for hub). Then, writing $v_{n+i} = v_i$, the wheel graph is $C_n = (V_C = \{v_i : i = 1, 2, \ldots, n\}, E_C = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, n\})$ (with vertex v_{n+1} being the same as vertex v_1); the spokes graph is $S_n = (V_C \cup \{v_0\}, E_S = \{(v_0, v_i) : i = 1, 2, \ldots, n\})$ and the Wheel-and-Spokes graph is the union of the wheel graph and the spokes graph $W_n = C_n \cup S_n = (V_C \cup \{v_0\}, E_C \cup E_S)$.



Wheel graph C_n Spokes graph S_n Wheel-and-Spokes graph W_n

Figure 1: Wheel, Spokes, and Wheel-and-Spokes graphs for $n = 6$

Taking a cue from Maiti and Sarkar (2019) who studied a SRW on the vertices of a circle (or a wheel without spokes) graph C_n, here we answer the following three questions about the SRW on a Wheel-and-Spokes graph W_n:

Q1) How long (or how many steps) does it take for the walk to go from any one vertex to another?

Q2) Starting from one vertex, how long does it take to visit all vertices (at least once)? Which vertex is visited the last?

Q3) Having visited all vertices, how many additional steps does it take to return to the starting vertex?

The answers to these questions are not numbers, but random variables for which we desire to obtain the exact probability mass functions (PMF), if possible. However, if it is too difficult to obtain the PMF, then we will settle for only their means and standard deviations. We compare our results to those of a SRW on the vertices on a circle.

Let the random time (number of steps) to go from vertex v_i to vertex v_j be called the transit time from i to j and be denoted by ${}_iT_j$ for $i \neq j = 0, 1, 2, \ldots, n$. Using the structural symmetry of the W_n graph, we establish the following equalities (in distribution):

1. (Hub to periphery): ${}_0T_1 = {}_0T_2 = {}_0T_3 = \ldots = {}_0T_n$
2. (Periphery to hub): ${}_1T_0 = {}_2T_0 = {}_3T_0 = \ldots = {}_nT_0$

3. (Periphery to periphery): ${}_iT_j = {}_jT_i = {}_{j+k}T_{i+k} = {}_{i+k}T_{j+k}$ for all $k \in \mathcal{Z}$ and for all $i \neq j = 1, 2, \ldots, n$, where addition of node labels is interpreted as a modulo n operation (however, v_n is the counter-clockwise neighbor of v_1 on the periphery, and $v_0 = H$ is the hub).

To develop the notion of time taken to return to the starting vertex, by suitable renumbering, we also define ${}_0T_0 = 1 + {}_1T_0$; and for $i = 1, 2, \ldots, n$, we define

$$ {}_iT_i = {}_1T_1 = \begin{cases} 1 + {}_2T_1 & \text{with probability } 2/3, \text{ and} \\ 1 + {}_0T_1 & \text{with probability } 1/3. \end{cases} $$

Along with the study of transit time from vertex v_i to vertex v_j, an associated question is: How many other vertices does the walk pass through? Let ${}_VP_j^t$ denote a path going through the set of points in V and ending in vertex v_j in t steps. We shall start with a singleton $V = \{v_i\}$, and then go through all sets of two vertices, then three vertices, and so on.

In Section 2, we study the mean and SD of transition time ${}_iT_j$ using a probabilistic method along with the expected number of vertices visited during the transition, while we study its PMF of ${}_iT_j$ in Section 3 using an enumeration method. In Section 4, we study the cover time; alongside, we also find the probability of the last visited vertex, and calculate the mean time to return to the starting vertex after visiting all vertices.

2 A Probabilistic Method for Mean and SD of ${}_iT_j$

In this section, we study the transition time ${}_iT_j$ from vertex v_i to vertex v_j. Here, we will focus on the mean and the SD of the random variable ${}_iT_j$, leaving the study of the PMF of ${}_iT_j$ for Section 3.

2.1 Time to Reach a Target Vertex

Let ${}_i\mu_j = E[{}_iT_j]$ denote the expected time for a path to travel from vertex i to vertex j. Let M denote the one-step Markov transition matrix for the walk on G, and let $\widetilde{M_j}$ be the adjusted Markov transition matrix for the graph $\widetilde{G}_j$, which is absorbing at v_j. In the latter situation, since the wheel-and-spoke graph exhibits rotational symmetry, it suffices to consider only two cases — $\widetilde{M_n}$ and $\widetilde{M_0}$ — corresponding to any vertex on the wheel or the vertex at the hub.

First, if v_0 at the hub is the absorbing vertex and the starting vertex is (without loss of generality) v_1 on the circle, then the walk reaches the hub in one step with probability 1/3, and with the remaining probability 2/3, the walk moves to an adjacent vertex v_2 or v_n. In this latter case, by renumbering the vertices on the periphery we can consider the current vertex is again v_1; that is, the process renews itself. Hence, ${}_1T_0$ is a geometric(1/3) random variable: In particular, $P\{{}_1T_0 = k\} = (2/3)^{k-1}(1/3)$ for $k = 1, 2, 3, \ldots$; and ${}_1T_0$ has mean 3 and variance 6.

Next, if v_n on the circle is the absorbing vertex, and the starting vertex is any other vertex, then writing the rows and the columns in the order corresponding to vertices $v_1, v_2, v_3, \ldots, v_{n-1}, H = v_0$, we write down the (i, j)-th element of $\widetilde{M_n}$ as follows.

$$\widetilde{m}_{i,j} = \begin{cases} 1/3 & \text{if } 1 \le i \le n-1 \text{ and } j = i-1, i+1, n \\ 1/n & \text{if } i = n \text{ and } 1 \le j \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

Note that this matrix is obtained by simply deleting the row and column corresponding to v_n in the ordinary MC on vertices $v_1, v_2, v_3, \ldots, v_{n-1}, v_n, v_0$. For instance, for graph W_5, the transition matrix for the ordinary MC on vertices $v_1, v_2, v_3, v_4, v_5, v_0$ is given by M_5, detailed below.

$$M_5 = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

If the stochastic process is absorbing at vertex v_5, then the transition matrix for the modified MC corresponds to vertices v_1, v_2, v_3, v_4, v_0, becoming the below matrix $\widetilde{M}_5$.

$$\widetilde{M}_5 = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

In order to find the properties of the random walk absorbing at vertex v_n, having transition matrix $\widetilde{M}_n$, following the method given in [2], we first find the inverse matrix $N = (I - \widetilde{M}_n)^{-1}$, known as the *fundamental matrix* of an absorbing Markov chain. This we do using the Gauss-Jordan elimination process, which proceeds in four steps.

Step 0: We start with the initial matrix $I - \widetilde{M}_n$, denoted by $\widetilde{N}^{(0)}$.

$$\widetilde{N}^{(0)} = \left[\begin{array}{ccccc|c} 1 & -\frac{1}{3} & 0 & \cdots & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} & \ddots & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 & \ddots & \vdots & -\frac{1}{3} \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{3} & \vdots \\ 0 & 0 & \cdots & -\frac{1}{3} & 1 & -\frac{1}{3} \\ \hline -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & 1 \end{array}\right] = \begin{bmatrix} \widetilde{N}^{(0)}_{1,*} \\ \vdots \\ \widetilde{N}^{(0)}_{n,*} \end{bmatrix}$$

Step 1: Keeping the first row and the n-th row of $\widetilde{N}^{(0)}$ unscathed, and proceeding from the second row to the $(n-1)$-st row one-by-one, we eliminate the entry $-\frac{1}{3}$ to the left of the main diagonal by adding to each row a suitable multiple of the previous row. When completed, we obtain $\widetilde{N}^(1)$,

$$\widetilde{N}^{(1)} = \left[\begin{array}{ccccc|c} d_1 & -\frac{1}{3} & 0 & \cdots & 0 & p_1 \\ 0 & d_2 & -\frac{1}{3} & \ddots & 0 & p_2 \\ 0 & 0 & d_3 & \ddots & \vdots & p_3 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{3} & \vdots \\ 0 & 0 & \cdots & 0 & d_{n-1} & p_{n-1} \\ \hline -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & 1 \end{array}\right] = \begin{bmatrix} \widetilde{N}^{(1)}_{0,*} \\ \vdots \\ \widetilde{N}^{(1)}_{n,*} \end{bmatrix} = \begin{bmatrix} \widetilde{N}^{(0)}_{1,*} \\ \widetilde{N}^{(0)}_{2,*} + \frac{1}{3d_1}\widetilde{N}^{(1)}_{1,*} \\ \vdots \\ \widetilde{N}^{(0)}_{n-1,*} + \frac{1}{3d_{n-2}}\widetilde{N}^{(1)}_{n-2,*} \\ \widetilde{N}^{(0)}_{n,*} \end{bmatrix}$$

where d_i and p_i satisfy the recursive relations

$$d_1 = 1,\ d_{i+1} = 1 - \frac{1}{9d_i},\ \text{and } p_1 = -\frac{1}{3},\ p_{i+1} = -\frac{1}{3} + \frac{1}{3d_i}p_i. \tag{1}$$

Lemma 1. For all $i \geq 1$, we have $d_i = \frac{F_{2i+2}}{3F_{2i}}$ and $p_i = \frac{1-F_{2i+1}}{3F_{2i}}$, where F_i is the i-th number of the Fibonacci sequence $(1, 1, 2, 3, 5, 8, \ldots)$.

Proof. The proof is by mathematical induction on i. Note that $d_1 = \frac{F_4}{3F_2}$. Assume that $d_i = \frac{F_{2i+2}}{3F_{2i}}$ holds. Then by the recursive relation (1), we have

$$d_{i+1} = 1 - \frac{1}{9d_i} = 1 - \frac{1}{9\frac{F_{2i+2}}{3F_{2i}}} = \frac{3F_{2i+2} - F_{2i}}{3F_{2i+2}} = \frac{F_{2i+4}}{3F_{2i+2}}.$$

Similarly, note that $p_1 = -\frac{1}{3} = \frac{1-F_3}{3F_2}$. Next, assume that $p_i = \frac{1-F_{2i+1}}{3F_{2i}}$ holds. Then by the recursive relation (1), we have

$$\begin{aligned} p_{i+1} &= -\frac{1}{3} + \frac{1}{3d_i}p_i = -\frac{1}{3} + \frac{1}{3\frac{F_{2i+2}}{3F_{2i}}}\frac{1-F_{2i+1}}{3F_{2i}} \\ &= -\frac{1}{3} + \frac{1-F_{2i+1}}{3F_{2i+2}} = \frac{1-F_{2i+1}-F_{2i+2}}{3F_{2i+2}} = \frac{1-F_{2i+3}}{3F_{2i+2}}. \end{aligned}$$

This completes the proof of the lemma.

Step 2: Keeping the last two rows of $\widetilde{N}^{(1)}$ unscathed, and proceeding backwards from the $(n-2)$-nd row to the first row one-by-one, we eliminate the entry $-\frac{1}{3}$ to the right of the main diagonal by adding to each row a suitable multiple of the following row. When completed, we obtain

$$\widetilde{N}^{(2)} = \left[\begin{array}{ccccc|c} d_1 & 0 & 0 & \cdots & 0 & q_1 \\ 0 & d_2 & 0 & \ddots & 0 & q_2 \\ 0 & 0 & d_3 & \ddots & \vdots & q_3 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & d_{n-1} & q_{n-1} \\ \hline -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & 1 \end{array}\right] = \begin{bmatrix} \widetilde{N}^{(2)}_{1,*} \\ \vdots \\ \widetilde{N}^{(2)}_{n,*} \end{bmatrix} = \begin{bmatrix} \widetilde{N}^{(1)}_{1,*} + \frac{1}{3d_2}\widetilde{N}^{(2)}_{2,*} \\ \vdots \\ \widetilde{N}^{(1)}_{n-2,*} + \frac{1}{3d_{n-1}}\widetilde{N}^{(2)}_{n-1,*} \\ \widetilde{N}^{(1)}_{n-1,*} \\ \widetilde{N}^{(0)}_{n,*} \end{bmatrix}$$

where q_i's are obtained recursively,

$$q_{n-1} = p_{n-1} = \frac{1-F_{2n-1}}{3F_{2n-2}},\ \text{and } q_i = p_i + \frac{q_{i+1}}{3d_{i+1}} = \frac{1-F_{2i+1}}{3F_{2i}} + \frac{F_{2i+2}}{F_{2i+4}}q_{i+1}.$$

Remark 1: By mathematical induction, one can obtain the following formula for q_i.

$$q_i = \frac{F_{2n}}{F_{2n-2i+4}}\left(1 + \frac{1}{3F_{2n}}\left(F_{2n+3} - F_{2n-2i+3} + \sum_{m=0}^{i-1}\frac{1-F_{2n-2m+3}}{F_{2n-2m}}\right)\right)$$

Step 3: Keeping rows 1 through $(n-1)$ unscathed, we reduce to zero all but the last entry in the last row by adding to the last row suitable multiples of rows 1 through $(n-1)$. When done, we get,

$$\widetilde{N}^{(3)} = \left[\begin{array}{ccccc|c} d_1 & 0 & 0 & \cdots & 0 & q_1 \\ 0 & d_2 & 0 & \ddots & 0 & q_2 \\ 0 & 0 & d_3 & \ddots & \vdots & q_3 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & d_{n-1} & q_{n-1} \\ \hline 0 & 0 & 0 & \cdots & 0 & D \end{array}\right] = \begin{bmatrix} \widetilde{N}^{(3)}_{1,*} \\ \vdots \\ \widetilde{N}^{(3)}_{n,*} \end{bmatrix} = \begin{bmatrix} \widetilde{N}^{(2)}_{1,*} \\ \vdots \\ \widetilde{N}^{(2)}_{n-1,*} \\ \widetilde{N}^{(0)}_{n,*} + \sum_{i=1}^{n-1}\frac{1}{nd_i}\widetilde{N}^{(2)}_{i,*} \end{bmatrix}$$

where D is the final element of the diagonal expressed as follows.

$$D = 1 + \sum_{i=1}^{n-1} \frac{q_i}{nd_i} = 1 + \sum_{i=1}^{n-1} \frac{3F_{2i}}{nF_{2i+2}} q_i$$

Step 4: We divide the last row by D. Thereafter, for each $i < n$, we subtract a q_i-multiple of the last row from row i to eliminate the i-th element of the n-th column, and then divide row i by the i-th diagonal element to normalize each row, yielding the identity matrix.

$$\widetilde{N}^{(4)} = I = \begin{bmatrix} \widetilde{N}^{(4)}_{1,*} \\ \vdots \\ \widetilde{N}^{(4)}_{n,*} \end{bmatrix} = \begin{bmatrix} \frac{1}{d_1}\widetilde{N}^{(2)}_{1,*} - \frac{q_1}{Dd_1}\widetilde{N}^{(3)}_{n,*} \\ \vdots \\ \frac{1}{d_{n-1}}\widetilde{N}^{(2)}_{n-1,*} - \frac{q_{n-1}}{Dd_{n-1}}\widetilde{N}^{(3)}_{n,*} \\ \frac{1}{D}\widetilde{N}^{(3)}_{n,*} \end{bmatrix}$$

The above-mentioned four steps complete the Gauss-Jordan elimination process. When these same four steps are applied to the identity matrix (in stead of the $\widetilde{N}^{(0)}$ matrix), we obtain the desired *fundamental matrix* of the absorbing Markov chain $N = [\widetilde{N}^{(0)}]^{-1} = (I - \widetilde{M_n})^{-1}$.

Parallel Step 0: We will apply the Gauss-Jordan elimination process starting with the initial matrix I, denoted by $N^{(0)}$, and given by

$$N^{(0)} = I = \begin{bmatrix} N^{(0)}_{1,*} \\ \vdots \\ N^{(0)}_{n,*} \end{bmatrix}$$

Parallel Step 1: We apply the same procedure as in Step 1, to obtain a lower triangular matrix $N^{(1)}$ as follows.

$$N^{(1)} = \left[\begin{array}{ccc|c} u_{1,1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] = \begin{bmatrix} N^{(1)}_{1,*} \\ \vdots \\ N^{(1)}_{n,*} \end{bmatrix} = \begin{bmatrix} N^{(0)}_{1,*} \\ N^{(0)}_{2,*} + \frac{1}{3d_1} N^{(1)}{}_{1,*} \\ \vdots \\ N^{(0)}_{n-1,*} + \frac{1}{3d_{n-2}} N^{(1)}{}_{n-2,*} \\ N^{(0)}_{n,*} \end{bmatrix}$$

Where $u_{i,j}$ is given recursively:

$$u_{i,j} = \begin{cases} \frac{u_{i-1,j}}{3d_{i-1}} & \text{if } j < i \\ 1 & \text{if } j = i \end{cases}$$

Lemma 2 For all $1 \le j \le i < n$, we have $u_{i,j} = \frac{F_{2j}}{F_{2i}}$.

Proof. Note that $u_{i,i} = 1 = \frac{F_{2i}}{F_{2i}}$. Assume that $u_{i,j} = \frac{F_{2j}}{F_{2i}}$. Then by the recursive definition, we obtain an explicit definition for $u_{i,j}$.

$$u_{i+1,j} = \frac{u_{i,j}}{3d_i} = \frac{1}{3\frac{F_{2i+2}}{3F_{2i}}} \frac{F_{2j}}{F_{2i}} = \frac{F_{2j}}{F_{2i+2}}$$

Parallel Step 2: Following the procedure in Step 2, we obtain $N^{(2)}$ as follows,

$$N^{(2)} = \left[\begin{array}{ccc|c} v_{1,1} & \cdots & v_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1,1} & \cdots & v_{n-1,n-1} & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] = \begin{bmatrix} N^{(2)}_{1,*} \\ \vdots \\ N^{(2)}_{n,*} \end{bmatrix} = \begin{bmatrix} N^{(1)}_{1,*} + \frac{1}{3d_2} N^{(2)}_{2,*} \\ \vdots \\ N^{(1)}_{n-2,*} + \frac{1}{3d_{n-1}} N^{(2)}_{n-1,*} \\ N^{(1)}_{n-1,*} \\ N^{(0)}_{n,*} \end{bmatrix}$$

where $v_{n-1,j} = u_{n-1,j} = \frac{F_{2j}}{F_{2n-2}}$, and, proceeding backwards, $v_{i,j}$'s are given by the recursive relations $v_{i-1,j} = \frac{v_{i,j}}{3d_i}$.

Lemma 3 For all $1 \leq i, j < n$, the entry $v_{i,j}$ is given by the following recursive definition.

$$v_{i,j} = \begin{cases} \frac{F_{2j}}{F_{2n-2}} & \text{if } i = n-1 \text{ and } j \leq n-1 \\ F_{2j}F_{2j+4} \sum_{k=j}^{n-1} \frac{1}{F_{2k}F_{2k+2}} & \text{if } 1 \leq i < n-1 \text{ and } 1 \leq j \leq i \\ \frac{F_{2j}}{F_{2i+4}} v_{j,j} & \text{if } 1 \leq i < n-1 \text{ and } i < j \leq n-1 \end{cases}$$

Parallel Step 3: Following the procedure in Step 3, we obtain $N^{(3)}$ as follows,

$$N^{(3)} = \left[\begin{array}{ccc|c} v_{1,1} & \cdots & v_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1,1} & \cdots & v_{n-1,n-1} & 0 \\ \hline w_1 & \cdots & w_{n-1} & 1 \end{array}\right] = \begin{bmatrix} N^{(3)}_{1,*} \\ \vdots \\ N^{(3)}_{n,*} \end{bmatrix} = \begin{bmatrix} N^{(2)}_{1,*} \\ \vdots \\ N^{(2)}_{n-1,*} \\ N^{(0)}_{n,*} + \sum_{i=1}^{n-1} \frac{1}{nd_i} N^{(2)}_{i,*} \end{bmatrix}$$

where $w_j = \sum_{i=1}^{n-1} \frac{v_{i,j}}{nd_i}$.

Parallel Step 4: Following the procedure in Step 4, we obtain the final matrix $N^{(4)} = (I - \widetilde{M_n})^{-1} = N$.

$$N^{(4)} = \left[\begin{array}{ccc|c} \frac{1}{d_1}\left(v_{1,1} - \frac{q_1}{D} w_1\right) & \cdots & \frac{1}{d_1}\left(v_{1,n-1} - \frac{q_1}{D} w_{n-1}\right) & -\frac{q_1}{Dd_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{1}{d_{n-1}}\left(v_{n-1,1} - \frac{q_{n-1}}{D} w_1\right) & \cdots & \frac{1}{d_{n-1}}\left(v_{n-1,n-1} - \frac{q_{n-1}}{D} w_{n-1}\right) & -\frac{q_{n-1}}{Dd_{n-1}} \\ \hline w_1 & \cdots & w_{n-1} & 1 \end{array}\right] = \begin{bmatrix} \frac{1}{d_1} N^{(2)}_{1,*} - \frac{q_1}{Dd_1} N^{(3)}_{n,*} \\ \vdots \\ \frac{1}{d_{n-1}} N^{(2)}_{n-1,*} - \frac{q_{n-1}}{Dd_{n-1}} N^{(3)}_{n,*} \\ \frac{1}{D} N^{(3)}_{n,*} \end{bmatrix}$$

Referring to Theorems 3.2.4, 3.3.3, and 3.5.7 in [2], using the matrix $N = [I - \widetilde{M_n}]^{-1}$ we get the following results:

1. The (x, y)-th entry of matrix N gives the expected number of times the walk starting at v_x visits v_y before being absorbed at v_j.

2. The expected number of steps before being absorbed when starting from vertex v_x is the x-th entry of the vector $\mathbf{t} = N\mathbf{1}$, where $\mathbf{1}$ is a column vector whose entries are all 1.

3. The variance of the number of steps before being absorbed when starting from v_x is the x-th entry of the vector $(2N - I)\mathbf{t} - \mathbf{t}_{\text{sq}}$, where $\mathbf{t}_{\text{sq}}$ is the Hadamard product of $\mathbf{t}$ with itself.

4. The (x, y)-th entry of matrix $K = (N - I)(N_{dg})^{-1}$, where N_{dg} is the diagonal of N, gives the probability that vertex v_y will be visited on the walk starting at v_x before being absorbed at v_j. The i-th row of $(K - K_{dg})1$, gives $E[{}_i\nu_j]$, the expected number of new vertices visited along the walk from v_i to v_j.

   We leave the expression for the standard deviation of ${}_i\nu_j$ to the reader.

2.2 Mean and SD of Transition Times Illustrated

Using the method of the previous subsection, here we illustrate some exact computations. For $n = 5$, we have the following results:

$$
\begin{aligned}
N = &\ \tfrac{1}{11} \begin{bmatrix} 18 & 12 & 9 & 6 & 15 \\ 12 & 24 & 15 & 9 & 20 \\ 9 & 15 & 24 & 12 & 20 \\ 6 & 9 & 12 & 18 & 15 \\ 9 & 12 & 12 & 9 & 25 \end{bmatrix} \\
\mathbf{t}_{\text{mean}} = &\ \tfrac{1}{11} \begin{bmatrix} 60 & 80 & 80 & 60 & 67 \end{bmatrix} \\
\mathbf{t}_{\text{var}} = &\ \tfrac{1}{121} \begin{bmatrix} 3990 & 4160 & 4160 & 3990 & 4124 \end{bmatrix} \\
\mathbf{t}_{\text{sd}} \approx &\ \tfrac{1}{11} \begin{bmatrix} 63.17 & 64.50 & 64.50 & 63.17 & 64.22 \end{bmatrix} \\
E[{}_i\nu_j] = &\ \tfrac{1}{120} \begin{bmatrix} 217 & 311 & 311 & 217 & 240 \end{bmatrix}
\end{aligned}
$$

For $n = 6$, we have:

$$
\begin{aligned}
N = &\ \tfrac{1}{40} \begin{bmatrix} 66 & 45 & 36 & 30 & 21 & 66 \\ 45 & 90 & 60 & 45 & 30 & 90 \\ 36 & 60 & 96 & 60 & 36 & 96 \\ 30 & 45 & 60 & 90 & 45 & 90 \\ 21 & 30 & 36 & 45 & 66 & 66 \\ 33 & 45 & 48 & 45 & 33 & 108 \end{bmatrix} \\
\mathbf{t}_{\text{mean}} = &\ \tfrac{1}{5} \begin{bmatrix} 33 & 45 & 48 & 45 & 33 & 39 \end{bmatrix} \\
\mathbf{t}_{\text{var}} = &\ \tfrac{1}{25} \begin{bmatrix} 1383 & 1485 & 1488 & 1485 & 1383 & 1470 \end{bmatrix} \\
\mathbf{t}_{\text{sd}} \approx &\ \tfrac{1}{5} \begin{bmatrix} 37.19 & 38.54 & 38.57 & 38.54 & 37.19 & 38.34 \end{bmatrix} \\
E[{}_i\nu_j] = &\ \tfrac{1}{792} \begin{bmatrix} 1693 & 2451 & 2624 & 2451 & 1693 & 1980 \end{bmatrix}
\end{aligned}
$$

For $n = 7$, we have:

$$
\begin{aligned}
N = &\ \tfrac{1}{29} \begin{bmatrix} 48 & 33 & 27 & 24 & 21 & 15 & 56 \\ 33 & 66 & 45 & 36 & 30 & 21 & 77 \\ 27 & 45 & 72 & 48 & 36 & 24 & 84 \\ 24 & 36 & 48 & 72 & 45 & 27 & 84 \\ 21 & 30 & 36 & 45 & 66 & 33 & 77 \\ 15 & 21 & 24 & 27 & 33 & 48 & 56 \\ 24 & 33 & 36 & 36 & 33 & 24 & 91 \end{bmatrix} \\
\mathbf{t}_{\text{mean}} = &\ \tfrac{1}{29} \begin{bmatrix} 224 & 308 & 336 & 336 & 308 & 224 & 277 \end{bmatrix} \\
\mathbf{t}_{\text{var}} = &\ \tfrac{1}{841} \begin{bmatrix} 70112 & 76622 & 77280 & 77280 & 76622 & 70112 & 76196 \end{bmatrix} \\
\mathbf{t}_{\text{sd}} \approx &\ \tfrac{1}{29} \begin{bmatrix} 264.79 & 276.81 & 277.99 & 277.99 & 276.81 & 264.79 & 276.04 \end{bmatrix} \\
E[{}_i\nu_j] = &\ \tfrac{1}{6864} \begin{bmatrix} 16847 & 24372 & 26629 & 26629 & 24372 & 16847 & 20592 \end{bmatrix}
\end{aligned}
$$

3 An Enumeration Method for the PMF of ${}_iT_j$

Although the probabilistic method of Section 2 suffices to give the mean and the variance of the number of steps of a random walk on W_n from vertex v_i to v_j, it is not sufficient to give the exact probability distribution of random walks on W_n. In this section, we consider an enumeration-based approach which is capable of determining the exact probability distribution of the transition time ${}_iT_j$ on W_n. In particular, using eigen decomposition of the adjacency matrix of the graph, we calculate the total number of paths of length k starting from vertex i which reach vertex j for the first time. We also calculate the total number of paths of length k starting from vertex v_i. The ratio of these two counts give the probability $P\{{}_iT_j = k\}$.

3.1 Eigen Decomposition of the Adjacency Matrix

For a simple random walk on an arbitrary graph G, let U denote the adjacency matrix of the graph G. Let $P^k = (p_1^k, \dots, p_n^k)$ be a column vector, where p_j^k denotes the number of paths which end at vertex v_j after k moves. Note that p_j^k is generic over the starting vertex, allowing for a general solution. We may see that $P^{k+1} = UP^k$. By repeated application of this relation, we obtain,

$$P^k = U^k P^0, \quad \text{for all} \quad k \geq 1 \tag{2}$$

where P^0 is the starting position, given by a vector of all zeros with the exception of a 1 in the i-th row, corresponding to the starting vertex v_i. Thus, the number of paths going from vertex v_i to vertex v_j in k steps, ${}_iP_j^k$, is given by the (j,i)-th element of U^k; that is, ${}_iP_j^k = (U^k)_{ji}$.

Furthermore, U, being the adjacency matrix of an undirected graph, is real and symmetric. Hence, U is diagonalizable (that is, there exist a matrix A whose columns are the eigenvectors $e_1, e_2, \dots, e_n$ of U, and a diagonal matrix B whose diagonal elements are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of U such that $U = ABA^{-1}$). For any diagonalizable graph, by repeated multiplication, we obtain an explicit formula for U^k in terms of its eigenvalues.

$$U^k = AB^kA^{-1} = A \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix} A^{-1}$$

Moreover, for the random walk on the Wheel-and-Spokes graph, with no absorbing state, the transition matrix U is real and symmetric. Consequently, the eigendecomposition ABA^{-1} of U involves an orthogonal matrix A, whence $U = ABA^T$. Thus, the set of symmetric matrices is a subset of all diagonalizable matrices, with the property that the A matrix is orthogonal only for a symmetric matrix. This simplifies the calculation of U^k, as $U^k = AB^kA^T$. Consequently, U^k is a linear combination of n rank-one matrices given by outer products of the eigenvectors $e_1, \dots, e_n$, with coefficients given by the k-th power of eigenvalues $\lambda_1, \dots, \lambda_n$. That is,

$$U^k = \sum_{i=1}^{n} \lambda_i^k (e_i e_i^T) \tag{3}$$

The above relation is the foundation of the following derivations specialized to the wheel-and-spokes graph W_n.

3.2 Number of Paths after k steps

Let the total number of paths which start at vertex v_i and end at some vertex on the wheel after k steps be denoted by ${}_iP_W^k$, and the total number of paths which start at vertex v_i and end at the hub after k steps be denoted ${}_iP_H^k$. Since every vertex on the wheel is connected to two neighboring vertices on the wheel and the hub, and since the hub is connected to every vertex on the wheel, we obtain the recursive definitions for ${}_iP_W^{k+1}$ and ${}_iP_H^{k+1}$.

$$
\begin{aligned}
{}_iP_W^{k+1} &= 2\ {}_iP_W^k + n\ {}_iP_H^k \\
{}_iP_H^{k+1} &= {}_iP_W^k
\end{aligned}
$$

Substituting the second equation into the first, we obtain a second order linear recurrence relation:

$$
{}_iP_W^{k+2} = 2\ {}_iP_W^{k+1} + n\ {}_iP_H^{k+1} = 2\ {}_iP_W^{k+1} + n\ {}_iP_W^k
$$

Solving the second order linear recurrence relation (see [1] for methodology), and writing $\alpha = 1 + \sqrt{n+1}, \beta = 1 - \sqrt{n+1}$, we obtain the following expressions:

$$
{}_iP_W^k = \begin{cases} \frac{n\left(\alpha^k - \beta^k\right)}{\alpha - \beta} & \text{if path starts at hub} \\ \frac{\sqrt{n+1}\left(\alpha^k + \beta^k\right) + \left(\alpha^k - \beta^k\right)}{\alpha - \beta} & \text{if path starts on wheel} \end{cases}
$$

Thereafter, we also obtain expressions for the number of paths along the wheel.

$$
{}_iP_H^k = \begin{cases} \frac{\sqrt{n+1}\left(\alpha^k + \beta^k\right) - \left(\alpha^k - \beta^k\right)}{\alpha - \beta} & \text{if path starts at hub} \\ \frac{\alpha^k - \beta^k}{\alpha - \beta} & \text{if path starts on wheel} \end{cases}
$$

Thus, the total number of paths of length k starting from vertex v_i (irrespective of the ending vertex) is obtained by adding ${}_iP_W^k$ and ${}_iP_H^k$, and is given by the proceeding equation.

$$
{}_iP_*^k = \begin{cases} \frac{\sqrt{n+1}\left(\alpha^k + \beta^k\right) + (n-1)\left(\alpha^k - \beta^k\right)}{\alpha - \beta} & \text{if path starts at hub} \\ \frac{\sqrt{n+1}\left(\alpha^k + \beta^k\right) + 2\left(\alpha^k - \beta^k\right)}{\alpha - \beta} & \text{if path starts on wheel} \end{cases}
$$

3.3 Number of Paths Reaching a Vertex for the First Time

Let ${}_iF_j^k$ denote the number of paths starting at v_i which visit v_j for the first time after k steps. Clearly, ${}_iF_j^k$ equals the number of paths from v_i to v_j *minus* the number of paths which have already visited v_j for the first time in $h < k$ steps, and have returned to v_j in an additional $k - h$ steps; that is,

$$
{}_iF_j^k = {}_iP_j^k - \sum_{h=0}^{k-1} {}_iF_j^h\, {}_jP_j^{k-h} \tag{4}
$$

where each ${}_xP_y^z$ may be calculated using (2) and (3).

Separating the last subtrahend in (4) and applying (4) on to ${}_iF_j^{k-1}$, we obtain a somewhat expanded equation.

$$
\begin{aligned}
{}_iF_j^k &= {}_iP_j^k - {}_jP_j^1\left({}_iP_j^{k-1} - \sum_{h=0}^{k-2} {}_iF_j^h\,{}_jP_j^{k-h-1}\right) - \sum_{h=0}^{k-2} {}_iF_j^h\,{}_jP_j^{k-h} \\
&= {}_iP_j^k - {}_jP_j^1\,{}_iP_j^{k-1} - \sum_{h=0}^{k-2} {}_iF_j^h\left({}_jP_j^{k-h} - {}_jP_j^1\,{}_jP_j^{k-h-1}\right)
\end{aligned}
$$

We will repeat this process of substitution for k iterations to obtain an expression for ${}_iF_j^k$. In order to generalize this process, we introduce the variables ν, S_ν, and the triangular array of coefficients $a_{\nu,0},\ldots,a_{\nu,k-\nu}$, with the initial values $S_1 = {}_iP_j^k$, $a_{1,h} = {}_jP_j^{k-h}$. With these substitutions, we obtain a simple equation for ${}_iF_j^k$.

$$
{}_iF_j^k = S_\nu - \sum_{h=0}^{k-\nu} {}_iF_j^h a_{\nu,h}
$$

Note that in the special case when $\nu = 1$, the above expression reduces to Equation (4). Performing the aforementioned procedure of expanding ${}_iF_j^k$, we obtain the following.

$$
\begin{aligned}
{}_iF_j^k &= S_\nu - \sum_{h=0}^{k-\nu} {}_iF_j^k a_{\nu,h} \\
&= S_\nu - a_{\nu,k-\nu}\,{}_iF_j^{k-\nu} - \sum_{h=0}^{k-(\nu+1)} {}_iF_j^h a_{\nu,h} \\
&= S_\nu - a_{\nu,k-\nu}\left({}_iP_j^{k-\nu} - \sum_{h=0}^{k-(\nu+1)} {}_iF_j^h\,{}_jP_j^{k-h-\nu-1}\right) - \sum_{h=0}^{k-(\nu+1)} {}_iF_j^h a_{\nu,h} \\
&= S_\nu - {}_iP_j^{k-\nu} a_{\nu,k-\nu} - \sum_{h=0}^{k-(\nu+1)} {}_iF_j^h\left(a_{\nu,h} - {}_jP_j^{k-h-\nu-1} a_{\nu,k-\nu}\right)
\end{aligned}
$$

From this, we derive the following recursive relations:

$$
\begin{aligned}
S_{\nu+1} &= S_\nu - {}_iP_j^{k-\nu} a_{\nu,k-\nu} \\
a_{\nu+1,h} &= a_{\nu,h} - {}_jP_j^{k-h-\nu-1} a_{\nu,k-\nu}
\end{aligned}
$$

Solving the above recursive relations, and noting that ${}_iF_j^k = S_k$, we obtain ${}_iF_j^k$.

$$
{}_iF_j^k = {}_iP_j^k + \sum_{\nu=1}^{k} (-1)^\nu\,{}_iP_j^{k-\nu} a_{\nu,k-\nu} \tag{5}
$$

Using Equation (5), we obtain the PMF of ${}_iT_j$ in terms of ${}_iF_j^k$ and ${}_iP_*^k$.

$$
{}_iT_j = k \ \text{ with probability } \ \frac{{}_iF_j^k}{{}_iP_*^k} \ \text{ for } \ k \geq 1
$$

Hence, the expected time the RW takes to go from vertex v_i to vertex v_j is as follows.

$$
E[{}_iT_j] = \sum_{k=0}^{\infty} k\,\frac{{}_iF_j^k}{{}_iP_*^k}
$$

Likewise, one can obtain the variance (or any other moment) of ${}_iT_j$. Illustrated numerical values for these values are already given in Subsection 2.2.

4 An Algorithmic Approach for the Cover Time

To calculate the expected cover time of the wheel-and-spokes graph, we use an algorithmic approach by repeatedly solving absorbing Markov chains. This technique also allows us to calculate the probability distribution of the last vertex visited.

Let a given walk $\mathbf{P}$ on the wheel-and-spokes graph be written as an ordered triplet (V, v_i, t), where V is the set of vertices that have been traveled through and v_i is the current vertex along the walk after t steps. Let us extend this notion of a walk to define a statistical walk $\overline{\mathbf{P}} = (V, v_j, t, p, \sigma^2)$, which has an associated probability p, variance σ^2, and which allows non-integer time t. This extended notion of a statistical walk $\overline{\mathbf{P}}$ will arise throughout the following section, and become a useful tool for representing statistical averages over all possible walks.

Let us define the neighborhood $N(V)$ of the vertex set V, which consists of all vertices not in V that are connected by an edge to some vertex in V. For instance, in Figure 2, with $V = \{1, 2, 3\}$, we have $N(V) = \{0, 4, 6\}$.

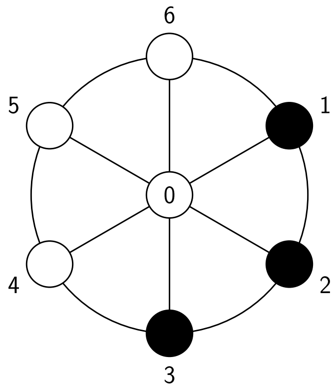


Figure 2: Illustrating a set of visited nodes V (dark) and its neighborhood $N(V)$

Next, let us define $L(\overline{\mathbf{P}})$ as the function yielding all $\overline{\mathbf{P'}} = (V', v_j', t', p', \sigma'^2)$ such that $v_j' \in N(V)$, $V' = V \cup \{v_j'\}, t' = t + \mu$, where μ is the expected absorption time, $p' = p\pi$, where π is the probability of being absorbed in vertex v_j', and $\sigma'^2 = \sigma^2 + \theta$, where θ is the variance of the absorption time. That is, $L(\overline{\mathbf{P}})$ consists all continuations $\overline{\mathbf{P'}}$ of the statistical walk $\overline{\mathbf{P}}$ such that $\overline{\mathbf{P'}}$ ends at a vertex not yet visited.

This enables us to define the collection $\mathcal{S}_k$ of all statistical walks consisting of k visited vertices; that is, $\mathcal{S}_k = \{\overline{\mathbf{P}} = (V, v_i, t, p, \sigma^2) : |V| = k\}$, where $\mathcal{S}_1 = \{\overline{\mathbf{P}}\}$ is the set containing only the starting point. We define $\mathcal{S}_{k+1} = \cup L(\overline{\mathbf{P}})\ \forall \overline{\mathbf{P}} \in \mathcal{S}_k$, as the set of all paths reachable from $\mathcal{S}_k$. Thus, to obtain the cover-time and statistical information about it, we only need to obtain $\mathcal{S}_{n+1}$, which will contain all $n+1$ vertices of the Wheel-and-Spokes graph W_n.

4.1 Solving the Restricted Absorbing Markov Chain

To find $\mathcal{S}_{n+1}$, we only need to obtain $L(\overline{\mathbf{P}})$. Thereafter, beginning with $\mathcal{S}_1$, which consists of the singleton starting vertex at time 0 with degenerate probability one on the starting vertex, we recursively obtain $\mathcal{S}_2$, $\mathcal{S}_3$, until we obtain $\mathcal{S}_{n+1}$. Proceeding as in Section 2, we construct an absorbing Markov chain to build the recursion.

Let $\overline{\mathbf{P}} = (V, v_i, t, p, \sigma^2)$ with $V = \{y_1, \dots, y_\eta\}$ and $v_i = y_h$ for some h. Also, let $N(V) = \{x_1, \dots, x_m\}$.

We focus on the restricted Markov chain on vertices $V, N(V)$, and proceed along the line in [2]. We write the one-step transition probabilities in a partitioned form as

$$\left[\begin{array}{c|c} Q_{\eta\times\eta} & R_{\eta\times n} \\ \hline 0 & I \end{array}\right]$$

where $q_{u,w}$ gives the one-step transition probability from a transient vertex y_u to another transient vertex y_w and $r_{u,w}$ gives the one-step transition probability from a transient vertex y_u to an absorbing vertex x_w. Let us write $N = (I - Q)^{-1}$.

Starting from the current vertex y_h, the probability of being absorbed at x_w is given by the row h and column w of the product matrix NR.

To obtain the expected number of steps before a path is absorbed at a particular absorbing vertex x_j, we consider a new absorbing Markov chain with transition matrix, Q_j, corresponding to the case where the transient states remain the same as V, but there is only one absorbing state x_j. In this case, the (u, w)'th element of Q_j gives the conditional one-step transition probability from transient vertex y_u to transient vertex y_w, given that the path is eventually absorbed at absorbing vertex x_j. This conditional probability, using Bayes' Theorem, is given as follows.

$$q^{(j)}_{u,w} = \frac{q_{u,w}(NR)_{w,j}}{\sum_z q_{u,z}(NR)_{z,j}} = q_{u,w}\,\frac{(NR)_{w,j}}{(NR)_{u,j}}$$

Recall that row/column h refers to vertex y_u and row/column j refers to vertex x_j. After obtaining $N_j = (I-Q_j)^{-1}$, we calculate the expected time $E[{}_hT_j]$ as the h'th element of N_j1, where 1 is a column vector of all ones (See [2]). Additionally, the variance on the number of steps $Var[{}_hT_j]$ may be calculated as h'th element of the column vector $t_{var} = (2N_j - I)t - t_{sq}$, where $t = N_j1$ for column vector 1 and t_{sq} gives the Hadamard product of t with itself. Having obtained both the expected time and the probability that a given path will be absorbed at a given neighbor, we may write out the full expression for L.

$$L((V, v_i, t, p, \sigma^2)) = \{(V \cup x_j, x_j, t + (N_j1)_{h,1}, p \times (NR)_{h,j}, \sigma^2 + (t_{var})_{h,1}) : x_j \in N(V)\}$$

4.2 The PMF of the Last Visited Vertex

Having obtained all elements of $\mathcal{S}_{n+1}$ using the technique mentioned in the previous subsection, if we add up the probabilities associated with those elements of $\mathcal{S}_{n+1}$ with the same final vertex visited, we obtain the PMF of the last vertex visited. Evaluating such quantities for the wheel-and-spoke graph for $n = 3, 4, \ldots, 11$, we draw the graph of the PMF of the last vertex visited in the next figure.

As n increases, $P\{L = 0\}$ deceases; that is, the hub becomes less and less likely to be the last vertex visited. This is quite natural since it is increasingly difficult to avoid the hub when at every step there is a 1/3 chance of visiting the hub. Also, the last vertex is equally likely to be as far on the wheel in the clockwise direction as in the counterclockwise direction and the corresponding probabilities increase gently the farther we go from the starting vertex in either direction. Hence, the last vertex is most likely to be the farthest from the starting vertex v_1. This result differs from that for the symmetric random walk on a circle where all vertices are equally likely to be the last!

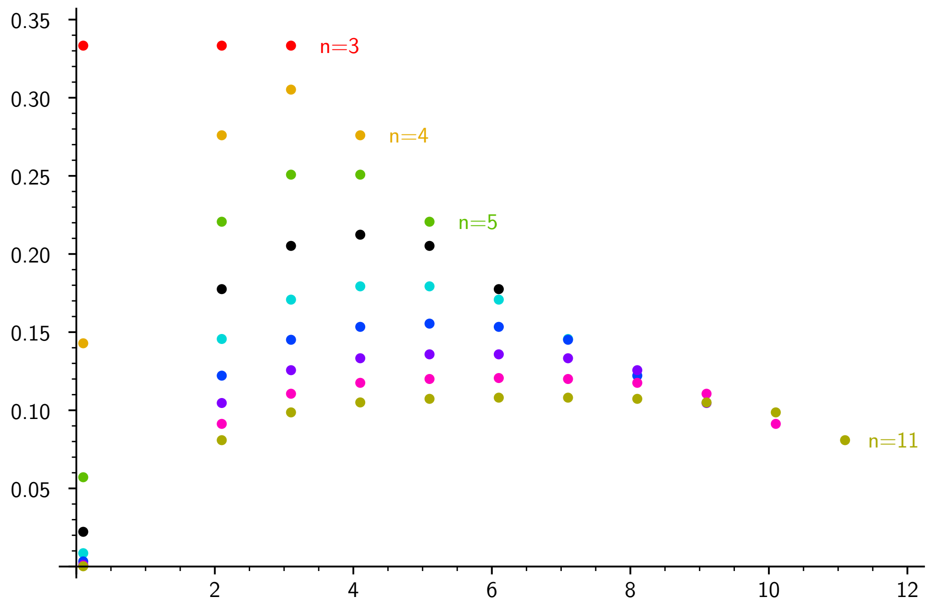


Figure 3: Probability $P_1[L = i]$ that the last vertex is v_i when starting at v_1

4.3 Expected Cover Time

The conditional expected cover time (given the last vertex visited) are shown in the next figure, and the overall expected cover time is listed in Table 1.

Obviously, as n increases, the conditional expected cover time (given the last vertex visited) also increase. If the hub happens to be the last vertex visited (with a low probability), then the conditional expected cover times are relatively smaller than if the last vertex is on the wheel, and the discrepancies increase with n. The conditional expected cover times are the same if the the last vertices are equally far from the starting vertex v_1 in the clockwise or the counterclockwise directions. The farther (in clockwise or counterclockwise direction) the last vertex from the starting vertex v_1, the longer (though not by much) it takes to visit all vertices.

Table 1: Expected cover times $E_0[\overline{T}], E_1[\overline{T}]$, if starting at hub or periphery, and for comparison $E_C[\overline{T}]$, the expected cover time of the Cycle graph

n	3	4	5	6	7	8	9	10	11
$E_1[\overline{T}]$	5.50	8.82	13.18	18.25	23.80	29.71	35.89	42.31	48.94
$E_0[\overline{T}]$	5.50	9.39	14.01	19.18	24.78	30.70	36.89	43.31	49.94
$E_C[\overline{T}]$	3	6	10	15	21	28	36	45	55

As $n \geq 7$ increases, the expected cover time starting from the hub is about one more than the expected cover time starting from the wheel. This is because starting from the wheel the hub is visited earlier than all vertices on the wheel (there is only a

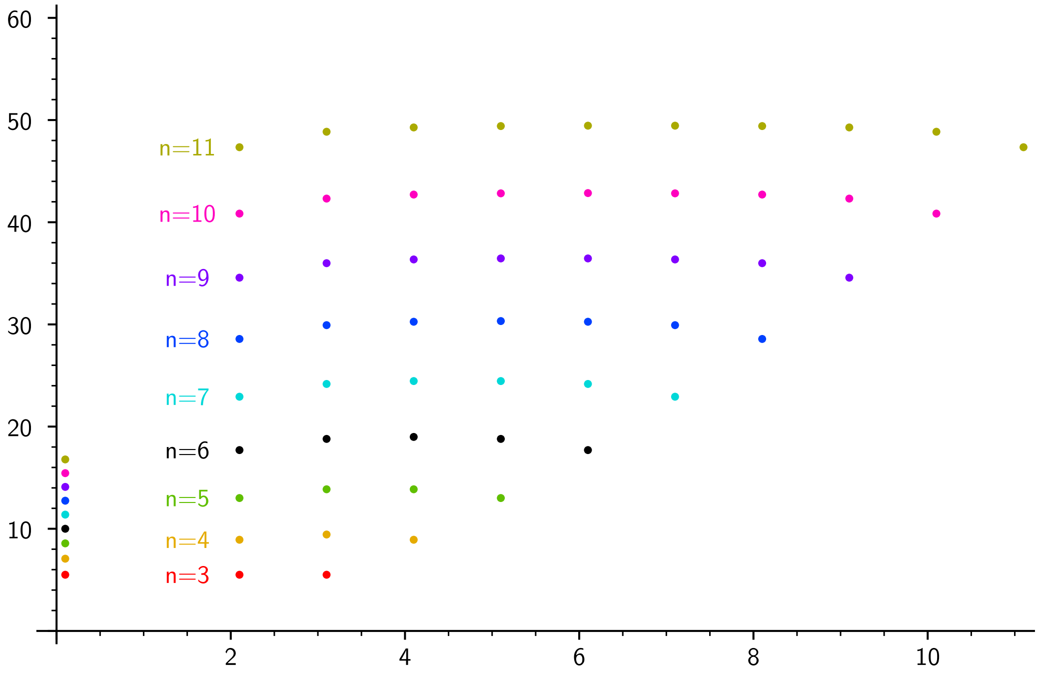


Figure 4: Conditional expected cover-time $E_1[\overline{T}|L = i]$ starting at v_1 and ending at v_i

negligible probability that the hub is the last vertex visited). Hence, starting from the hub essentially means that the first step is an extra step wasted to go to the periphery; thereafter, the walk is similar to the one that starts on the wheel.

Notably, compared to the expected cover time for the Cycle graph studied in [4], the Wheel-and-Spokes graph takes longer up until $n = 9$ for $E_1[\overline{T}]$ and $n = 10$ for $E_0[\overline{T}]$. After this point the Wheel-and-Spokes graph begins to take less time than the Cycle graph, because the Wheel-and-Spokes graph giving the random walks more opportunities to visit yet-unvisited vertices by going through the hub. For small n these visits through the hub tend to waste steps, causing the Wheel-and-Spokes graph to be less efficient.

4.4 Return Time

Here we evaluate $E[{}_LT_i]$, the expected time to return to the starting vertex after visiting all vertices.

As seen in Section 2, when the starting vertex is the hub; that is, $i = 0$, then irrespective of the final vertex L, the return time to the hub is a geometric random variable with success probability $\frac{1}{3}$. Hence, the expected return time is 3 with a variance of 6.

On the other hand, if the starting vertex is i on the wheel, then we can find the expected return time after visiting all vertices, by combining the results of the previous subsection and Section 2. We simply take the weighted average of the conditional expected return time from each possible last vertex L to the starting vertex i with weights given by the probability of each last vertex L. Thus $E[{}_LT_i]$ is given by the following equation.

$$E[{}_LT_i] = \sum_{j \neq i} P[L = j] \; E[{}_jT_i]$$

This equation may also be expressed in a more succinct form using linear algebra. In fact, without loss of generality, we can assume that the starting vertex on the wheel is $i = n$. Let $\mathbf{P}$ be the column vector of probabilities that the last vertex L equals vertices $v_1, v_2, v_3, \ldots, v_{n-1}, v_0$, respectively. Using $\mathbf{t}_{mean}$ from Section 2, which gives the mean time to travel from v_j to v_n (for $j = 1, 2, \ldots, n-1, 0$), we obtain the following equation for $E[_LT_n]$.

$$E[_LT_n] = \mathbf{P}^T\mathbf{t}_{mean}$$

The proceeding table gives the expected return time to the starting vertex on the wheel after visiting all vertices for $3 \leq n \leq 11$.

Table 2: Expected return time $E[_LT_n]$

n	3	4	5	6	7	8	9	10	11
$E[_LT_n]$	3.00	4.62	6.40	8.25	10.11	11.98	13.84	15.68	17.52

Clearly, for $n = 3$, the return time is a geometric random variable with success probability 1/3. Hence, $E[_LT_n] = 3$. Thereafter, as n increases, the expected return time to the starting vertex increases.

5 Conclusion

In this paper, we have obtained several results pertaining to random walks on the Wheel-and-Spokes graph — the expectation and distribution of the time to proceed from one vertex to another, of the time to visit all vertices, and of the number of steps taken to return to the starting vertex.

To obtain these question we relied on three distinct methods — probabilistic, enumeration and algorithmic — each relying on linear algebraic interpretations of the problem.

This research may allow for the modeling of certain real-world events. The study of random walks on the Wheel-and-Spokes graph is useful, for example, to model the spread of information with a centralized source, such as a public forum. Information might be posted to the forum, where it can be seen by any user, or might spread by word of mouth between neighbors, mirroring the path of information to/from the hub or along the wheel respectively. Future research might consider an asymmetry in the probability of traveling to the hub compared to traveling along the wheel or in the probability of traveling back on an edge just used and other adjacent edges. Another direction of future research is to change the graph. For instance, the hub may be replaced or augmented by a separate inner circle, to represent different channels of communication, or multiple connected wheels could be overlaid to form a spiderweb-like graph.

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